Stable directions for small nonlinear Dirac standing waves

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Abstract: We prove that for a Dirac operator, with no resonance at thresholds nor eigenvalue at thresholds, the propagator satisfies propagation and dispersive estimates.
When this linear operator has only two simple eigenvalues sufficiently close to each other, we study an associated class of nonlinear Dirac equations which have stationary solutions. As an application of our decay estimates, we show that these solutions have stable directions which are tangent to the subspaces associated with the continuous spectrum of the Dirac operator. This result is the analogue, in the Dirac case, of a theorem by Tsai and Yau about the Schrödinger equation. To our knowledge, the present work is the first mathematical study of the stability problem for a nonlinear Dirac equation.

Introduction

We study the stability of stationary solutions of a time-dependent nonlinear Dirac equation.
Usually, a localized stationary solution of a given time-dependent equation represents the bound state of a particle. Like Ranada [Ran], we call it a *particle like solutions* (PLS). In the literature, the term *soliton* is also found instead of PLS, but this additionally means that the particle keeps its form after a collision. Many works have been devoted to the proof of the existence of such solutions for a large variety of equations. Although their stability is a crucial problem (in particular in numerical computation or experiment), a smaller attention has been deserved to this issue.
There are different definitions of stability. The first one is commonly called the *orbital stability*. It means that the orbit of the perturbation of a PLS stays close to the PLS or a manifold of PLS but does not necessarily converge. A stronger notion is *asymptotic stability*, which means that the perturbation of the PLS relaxes asymptotically towards a PLS which is not far from the perturbed PLS. In fact in many conservative problems asymptotic stability does not hold. But one has asymptotic stability for a restricted class of perturbations, forming the so-called *stable manifold*.

In this paper, we deal with the problem of stability of small PLS of the following nonlinear Dirac equation:

$$\imath\partial_t\psi = (D_m + V)\psi + \nabla F(\psi) \qquad \text{(NLDE)}$$

where ∇F is the gradient of $F : \mathbb{C}^4 \mapsto \mathbb{R}$ for the standard scalar product of $\mathbb{R}^8$. Here, D_m is the usual Dirac operator [Tha92] acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$D_m = \alpha \cdot (-\imath\nabla) + m\beta = -\imath \sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

where $m \in \mathbb{R}_+^*$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are $\mathbb{C}^4$ hermitian matrices satisfying the following properties:

$$\begin{cases} \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} \mathbf{1}_{\mathbb{C}^4}, & i, k \in \{1, 2, 3\}, \\ \alpha_i \beta + \beta \alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1, 2, 3\}, \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}. \end{cases}$$

Here we choose

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$
$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In (NLDE), V is the external potential field and $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a nonlinearity such that

$$\forall (\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta} z) = F(z).$$

Some additional assumptions on F and V will be made in the sequel. Stationary solutions (PLS) of (NLDE) take the form $\psi(t, x) = e^{-\imath Et}\phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{PLSE}$$

We prove the existence of a manifold of small solutions to (PLSE), interpreted as particle like solutions to (NLDE). Then we construct a stable manifold around this manifold. At the origin, it is tangent to the sum of the eigenspace associated with the first eigenvalue and the continuous spectral subspace of $D_m + V$. This is the analogue in the Dirac case of [TY02d, Theorem 1.1, non-resonant case]. The interpretation is that radiations (described by the continuous spectrum) do not destabilize too much the PLS manifold. To prove stabilization towards the PLS manifold, we shall need linear decay estimates associated with the continuous spectral subspace of $D_m + V$.

To our knowledge, this is the first stability result on a nonlinear Dirac equation.

The problem of stability has been extensively studied for Schrödinger and Klein-Gordon equations. The methods used to treat these cases cannot be easily adapted to our problem, due to the fact that the Dirac operator D_m is not bounded-below, contrarily to $-\Delta$. The non-negativity of the latter permits to use minimization and concentration-compactness methods to prove the existence of orbitally stable standing waves, see e.g. Cazenave and Lions [CL82] or more recently Cid and Felmer [CF01].

In his review on nonlinear Dirac models, Ranada [Ran] writes that physicists first claimed that PLS (Particle Like Solutions) of the nonlinear Dirac equation couldn't be stable since the second derivative of the energy functional is

not positive-definite. Actually, in a very general setting (not related to the Dirac case), Shatah and Straus [SS85] and Grillakis, Shatah and Straus [GSS87] proved a general orbital stability condition even if the hessian of the energy functional is not positive-definite. Their conditions allow only one simple negative eigenvalue (and a kernel of dimension one also) for the second variation. It therefore cannot be directly applied to the Dirac case. However, it gave rise to an interesting discussion about the application of this method to the Dirac equation in some physical papers [SV86, AS86, BSV87]. Ranada also refers to numerical experiments which seem to confirm that some PLS are asymptotically stable in the Dirac case.

In the Schrödinger case, the asymptotic stability has been extensively studied during the last decade. A fundamental work is the one of Soffer and Weinstein [SW90, SW92], which is devoted to the study of a small nonlinear perturbation of a Schrödinger operator having one simple eigenvalue. They proved that the perturbed small PLS relaxes to a PLS. Later, Pillet and Wayne [PW97] proposed a different proof in the spirit of the central manifold theorem. In all these works, asymptotic stability is a direct consequence of propagation or dispersive estimates on the Schrödinger operator. In order to be able to use these estimates, one has to to consider the initial state (at time $t = 0$) of the perturbation as localized *i.e.* in L^1 or in L^2 weighted spaces with growing weight. To avoid such an assumption, Gustafson, Nakanishi and Tsai [GNT04] proposed to use Strichartz estimates.

Generalizations have been considered for instance by Tsai and Yau [TY02a, TY02c, TY02d, TY02b, Tsa03], who treated the case of a Schrödinger operator having two simple eigenvalues. An interesting phenomenon appeared: if the two eigenvalues are sufficiently distant one from the other, then after linearization around the excited state, one obtains a resonance. Tsai and Yau showed that if there is no resonance, the manifold of ground state has stable directions. In the resonant case, the manifold of ground states is asymptotically stable, whereas the manifold of excited states has stable and unstable directions (in case of instability, under some conditions, one has relaxation to the ground state). For a similar result, see also [SW04, SW05]. Notice that earlier Soffer and Weinstein [SW99] studied a similar resonance phenomenon in the case of the Klein-Gordon equation with a simple eigenvalue; they showed that it induced “metastability”. Another problem has been studied by Cuccagna [Cuc01, Cuc03, Cuc05]. He considered the case of big PLS, when the linearized operator has only one eigenvalue and obtained the asymptotical stability of the manifold of ground states. Tsai, Yau and Cuccagna also need propagation or dispersive estimates. The latter is proved by generalizing the work of Yajima [Yaj95] on wave operator.

Interesting development are also given by Rodnianski, Schlag and Soffer [RSS05a] who proved asymptotic stability of an arbitrary number of weakly interacting big PLS. Schlag [Sch04] and Krieger and Schlag [KS05] proved the existence of stable direction for unstable big PLS. We point out that some of the works of Schlag [ES04, GS04, RSS05b] or Soffer [HSS99, JSS91, RSS05b] are dedicated to prove dispersive estimates.

We also would like to mention the works of Buslaev and Perel'mann [BP95, BP92b, BP92c, BP92a], Buslaev and Sulem [BS03, BS02] or Weder [Wed00], in the one dimensional Schrödinger case.

Here, we study a nonlinear Dirac equation as a perturbation of a linear Dirac equation with a Dirac operator possessing only two simple eigenvalues sufficiently close to each other. Hence, we avoid problems of resonance after linearization around a PLS. The paper is organized as follows.

In section 1, we define the important objects and state our main results. We start with the propagation and dispersive linear estimates which will be crucial tools for this study. Then, we consider the nonlinear equation (NLDE) and state the existence of the PLS manifold. Eventually, we present our main theorem in which the stable manifold is constructed.

The section 2 is devoted to the proof of the propagation estimate, which uses spectral techniques. This is a time decay estimate in weighted L^2 spaces, expressing the fact that states associated with the continuous spectrum are not stationary. We use Mourre estimate similarly to Hunziker, Sigal and Soffer [HSS99] (for a generalization of the method, see e.g. [BdMGS96]). This method cannot be used in the neighborhood of the thresholds which needs a specific treatment. In particular, problems can occur in the presence of eigenvalues at thresholds or resonances at thresholds, and we shall assume in the whole paper that we are not in this situation. For the Schrödinger case, a similar problem has been studied by Jensen and Kato [JK79], Jensen and Nenciu [JN01, JN04]. Our arguments near the thresholds are inspired of these works. For a related study, see the article of Fournais and Skibsted [FS04] dealing with long range perturbations of Schrödinger operators.

In Section 3, we then prove the dispersive estimate, using the propagation estimate established in Section 2. For an interesting survey on dispersive estimates for Schrödinger operators, see Schlag [Sch05]. We have not been able to generalize the methods used in the Schrödinger case, in fact it seems that the Dirac equation with a potential behaves like a Klein-Gordon equation with a magnetic potential. This fact has already been noticed by D'Anconna and Fanelli in [DF], where they proved simultaneously dispersive estimates for a massless Dirac equation with a potential and for a wave equation with a magnetic potential. Our method is here inspired of the work by Cuccagna and Schirmer [CS01].

Finally, the last sections are devoted to the proof of our main result concerning the stability of the stationary solutions of (NLDE). We assume that the Dirac operator $D_m + V$ have only two simple eigenvalues and that it has no eigenvalues at thresholds nor resonances at thresholds. Note that our assumptions exclude electric potentials, for which the theorem of Kramers states that the eigenvalues are always degenerate, see [Par90, BH92]. In Section 4, this permits us to construct a manifold of PLS and then to study the spectrum of the linearized operator. This in turn, in Section 5, will allow us to decompose a solution of (NLDE) in three parts: the PLS part, the dispersive part associated with the continuous spectrum and a part corresponding to "excited states". This last part needs a particular treatment since it is not dispersive and hence disturbs the relaxation towards the PLS manifold.

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1. Main results

This section is devoted to the presentation of the model and the statement of our main results.

1.1. Decay estimates for a Dirac operator with potential. Let us first state our results concerning the time decay of $e^{-\imath t(D_m+V)}$ in weighted L^2 spaces and Besov spaces. This kind of estimates are called respectively propagation and dispersive estimates. As mentioned in the introduction, these results will be very important tools for the study of our nonlinear time-dependent Dirac equation.

The following spaces will be needed to state the main result of this subsection.

Definition 1.1 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3,\mathbb{C}^4)=\left\{f\in\mathcal{S}'(\mathbb{R}^3),\ \|\langle Q\rangle^\sigma\langle P\rangle^t f\|_2<\infty\right\}$$

for $\sigma,t\in\mathbb{R}$. We endow it with the norm

$$\|f\|_{H^t_\sigma}=\|\langle Q\rangle^\sigma\langle P\rangle^t f\|_2.$$

If $t=0$, we write L^2_σ instead of H^0_σ.

We have used the usual notations $\langle u\rangle=\sqrt{1+u^2}$, $P=-i\nabla$, and Q is the operator of multiplication by x in $\mathbb{R}^3$. For the sake of clarity, let us also recall the

Definition 1.2 (Besov space). *For $s\in\mathbb{R}$ and $1\le p,q\le\infty$, the Besov space $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^4)$ is the space of all $f\in\mathcal{S}'(\mathbb{R}^3,\mathbb{C}^4)$ (dual of the Schwartz space) such that*

$$\|f\|_{B^s_{p,q}}=\left(\sum_{j\in\mathbb{N}}2^{jsq}\|\varphi_j*f\|_p^q\right)^{\frac{1}{q}}<+\infty$$

with $\widehat{\varphi}\in\mathcal{D}(\mathbb{R}^n\setminus\{0\})$ such that $\sum_{j\in\mathbb{Z}}\widehat{\varphi}(2^{-j}\xi)=1$ for all $\xi\in\mathbb{R}^3\setminus\{0\}$, $\widehat{\varphi}_j(\xi)=\widehat{\varphi}(2^{-j}\xi)$ for all $j\in\mathbb{N}^*$ and for all $\xi\in\mathbb{R}^3$, and $\widehat{\varphi_0}=1-\sum_{j\in\mathbb{N}^*}\widehat{\varphi}_j$. We endow it with the norm $f\in B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^4)\mapsto\|f\|_{B^s_{p,q}}$.

In the whole chapter, we shall work within the following

Assumption 1.1. *The potential $V:\mathbb{R}^3\mapsto S_4(\mathbb{C})$ (self-adjoint 4×4 matrices) is a $\mathcal{C}^\infty$ function such that there exists $\rho>5$ with*

$$\forall\alpha\in\mathbb{N}^3,\ \exists C>0,\ \forall x\in\mathbb{R}^3,\ |\partial^\alpha V|(x)\le\frac{C}{\langle x\rangle^{\rho+|\alpha|}}.$$

Notice that by the Kato-Rellich Theorem , the operator

$$H:=D_m+V$$

is essentially self-adjoint on $\mathcal{C}^\infty_0(\mathbb{R}^3,\mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3,\mathbb{C}^4)$. We also work with the

Assumption 1.2. *The operator H presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is an eigenvector in $H^{1/2}_{-\sigma}(\mathbb{R}^3,\mathbb{C}^4)\setminus H^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ for some $\sigma\in(1/2,\rho-1/2)$ here. Let

$$\mathbf{P}_c(H)=\mathbb{1}_{(-\infty,-m]\cup[+m,+\infty)}(H) \tag{1.1}$$

be the projector associated with the continuous spectrum of H and

$$\mathcal{H}_c=\mathbf{P}_c(H)L^2(\mathbb{R}^3,\mathbb{C}^4). \tag{1.2}$$

We are now able to state our

Theorem 1.1 (Propagation for perturbed Dirac dynamics). *Assume that Assumptions 1.1 and 1.2 hold and let be $\sigma>5/2$. Then one has*

$$\|e^{-\imath tH}\mathbf{P}_c\,(H)\,\|_{B(L^2_\sigma,L^2_{-\sigma})}\le C\,\langle t\rangle^{-3/2}\,.$$

The proof of this result will be given in Section 2. We notice that it is still true if we assume $\rho>3$ in Assumption 1.1.
Our next result is the following theorem, proved in Section 3.

Theorem 1.2 (Dispersion for perturbed Dirac dynamics). *Assume that Assumptions 1.1 and 1.2 hold. Then for $p\in[1,2]$, $\theta\in[0,1]$, $s-s'\ge(2+\theta)(\frac{2}{p}-1)$ and $q\in[1,\infty]$ there exists a constant $C>0$ such that*

$$\|e^{-\imath tH}\mathbf{P}_c(H)\|_{B^s_{p,q},B^{s'}_{p',q}}\le C\,(K(t))^{\frac{2}{p}-1}$$

with $\frac{1}{p}+\frac{1}{p'}=1$, and

$$K(t)=\begin{cases}|t|^{-1+\theta/2} \text{ if } |t|\in(0,1],\\[2mm] |t|^{-1-\theta/2} \text{ if } |t|\in[1,\infty).\end{cases}$$

1.2. The stable manifold around the PLS for the nonlinear Dirac equation. We now want to study the following nonlinear Dirac equation

$$\begin{cases}\imath\partial_t\psi=H\psi+\nabla F(\psi)\\ \psi(0,\cdot)=\psi_0.\end{cases} \tag{1.3}$$

with $\psi\in\mathcal{C}^1(I,H^1(\mathbb{R}^3,\mathbb{C}^4))$ for some open interval I which contains 0 and where we recall that $H=D_m+V$. The nonlinearity $F:\mathbb{C}^4\mapsto\mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$ and hence the ∇ symbol has to be understood for the real structure of $\mathbb{C}^4$. For the usual hermitian product of $\mathbb{C}^4$, one has

$$DF(v)h=\Re\langle\nabla F(v),h\rangle.$$

We work within the following

Assumption 1.3. *The operator H has only two simple eigenvalues $\lambda_0 < \lambda_1$, with ϕ_0 and ϕ_1 as associated normalized eigenvectors. Moreover, the non resonant condition*

$$|\lambda_1 - \lambda_0| < \min\{|\lambda_0 + m|, |\lambda_0 - m|\}$$

holds.

Assumption 1.4. *The function $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is in $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$, is a homogeneous polynomial of degree 4 (i.e. with $D^\alpha F(z) = 0$ for $|\alpha| = 5$ and $D^\beta F(0) = 0$ for $|\beta| \leq 4$) or satisfies $F(z) = O(|z|^5)$ as $z \to 0$. Moreover, it has the gauge invariance property:*

$$F(e^{i\theta} z) = F(z), \forall z \in \mathbb{C}^4, \ \forall \theta \in \mathbb{R}.$$

We will prove in Theorem 1.3 that some solutions of the equation (1.3) are global and can be decomposed as the sum of a PLS plus a remainder part which is vanishing. Since the PLS part may change during the evolution, we need to track it. So we prove that around the origin, PLS form a manifold. We have the

Proposition 1.1 (PLS manifold). *Suppose that Assumptions 1.1–1.4 hold. Then for any $\sigma \in \mathbb{R}^+$, there exists Ω a neighborhood of $0 \in \mathbb{C}$, a $\mathcal{C}^\infty$ map*

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^\infty$ map $E : \Omega \mapsto \mathbb{R}$ such that $S(u) = u\phi_0 + h(u)$ satisfy for all $u \in \Omega$,

$$HS(u) + \nabla F(S(u)) = E(u)S(u), \tag{1.4}$$

with the following properties

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

Proof. This kind of results is now classical and left to the reader. For more details, see Subsection 4.1.

We are now able to write the main theorem of this paper. Its proof is given in Section 6. To state it we need the space $\mathcal{H}_c$ defined in 1.2.

Theorem 1.3 (Stable manifold). *Suppose that Assumptions 1.1–1.4 hold. Let $s, s', \beta \in \mathbb{R}_+^*$ be such that $s' \geq s + 3 \geq \beta + 6$ and $\sigma > 5/2$. There exists $\varepsilon_0 > 0$, $R > 0$, $K > 0$ and a Lipshitz map*

$$\Psi : B_{\mathbb{C}}(0, \varepsilon) \times \left(\mathcal{H}_c \cap B_{H^{s'}_\sigma}(0, R)\right) \mapsto \mathbb{C}$$

with $\Psi(v, 0) = 0$,

$$|\Psi(v, \xi)| \leq K \left(|v| + \|\xi\|_{H^{s'}_\sigma}\right)^2,$$

and such that the following hold. For any initial condition of the form

$$\psi_0 = S(v_0) + \xi_0 + \Psi(v_0, \xi_0)\phi_1$$

with $v_0 \in B_{\mathbb{C}}(0, \varepsilon)$ and $\xi_0 \in \mathcal{H}_c \cap B_{H^{s'}_\sigma}(0, R)$, one has

(i) there exists a unique global solution ψ of (1.3) *in*

$$\mathcal{C}^\infty\left(\mathbb{R}, H^{s'}(\mathbb{R}^3,\mathbb{C}^4)\cap H^s_{-\sigma}(\mathbb{R}^3,\mathbb{C}^4)\cap B^\beta_{\infty,2}(\mathbb{R}^3,\mathbb{C}^4)\right);$$

(ii) there exists $(v_\infty;\xi_\infty;E_\infty)\in\mathbb{C}\times H^{s'}_\sigma\cap\mathcal{H}_c\times\mathbb{R}$ *with*

$$|v_\infty - v_0|\le K\|\xi_0\|^2_{H^{s'}_\sigma},\quad |E_\infty|\le K\|\xi_0\|^2_{H^{s'}_\sigma},\quad \|\xi_\infty-\xi_0\|_{H^{s'}}\le K\|\xi_0\|^2_{H^{s'}_\sigma}$$

such that

$$\psi(t) = e^{-\imath(tE(v_\infty)+E_\infty)}S(v_\infty)+e^{-\imath tH}\xi_\infty+\varepsilon(t),$$

where

$$\begin{cases}\|\varepsilon(t)\|_{H^{s'}}\le K\|\xi_0\|_{H^{s'}_\sigma}\\ \|\varepsilon(t)\|_{H^s_{-\sigma}}\le \dfrac{K}{\langle t\rangle^2}\|\xi_0\|_{H^{s'}_\sigma}\\ \|\varepsilon(t)\|_{B^\beta_{\infty,2}}\le \dfrac{K}{\langle t\rangle^2}\|\xi_0\|_{H^{s'}_\sigma}.\end{cases}$$

as $t\to+\infty$.

Remark 1.1. The proof of these theorem work also if we want to obtain an expansion of the form

$$\psi(t)=e^{-\imath(tE(v_\infty)+E_\infty)}S(v_\infty)+e^{-\imath tD_m}\widetilde{\xi_\infty}+\varepsilon(t)$$

with the free Dirac operator. But in this case, we only have the estimates

$$\left\|\widetilde{\xi}_\infty-\xi_0\right\|_{H^{s'}}\le K\|\xi_0\|_{H^{s'}_\sigma}$$

see the remark following Lemma 5.10.

We notice that the stabilization is "faster" than the propagation and the dispersion: it is of order $\langle t\rangle^{-2}$ whereas $e^{-\imath tH}\xi_\infty$ is of order $\langle t\rangle^{-3/2}$ by Theorems 1.1 and 1.2. Hence the theorem states the existence of a family of initial states which form a manifold tangent at the origin to the sum of the eigenspace of H associated with λ_0 and the subspace associated with the continuous spectrum of H: $\mathcal{H}_c$. This family of initial states gives rise to solutions of (1.3) which asymptotically split in two parts. The first one is a PLS: $e^{-\imath(tE(u_\infty)+E_\infty)}S(u_\infty)$ the other is a dispersive perturbation: $e^{-\imath tH}\xi_\infty$. Hence if one perturbs a PLS in the direction of the continuous spectrum then this PLS relaxes to another PLS by emitting a dispersive wave.
This phenomenon is due to the propagation and the dispersion properties of the subspace associated with the continuous spectrum of H. We don't think that such a phenomenon could take place for perturbations in the direction of the excited states ϕ_1. Indeed, on this subspace, the dynamic seems to be conservative. The fact that we use propagation and dispersive estimates restricts the family of perturbations to regular and localized ones.

We now turn to the proof of our results.

2. Proof of Theorem 1.1: propagation estimates

Here we prove the propagation estimates of Theorem 1.1. The method used by Jensen and Kato [JK79] to prove this kind of estimates for Schrödinger operator works only for initial states which are spectrally localized near the thresholds $\pm m$. They used the spectral density as the Fourier transform of the propagator. But the Dirac resolvent

$$R_V(\lambda \pm \imath\varepsilon) = (H - \lambda \mp \imath\varepsilon)^{-1}$$

does not decay in $B(L^2_\sigma, L^2_{-\sigma})$ as $|\lambda| \to +\infty$ for any $\sigma > 0$, see [Yam93]. So we cannot use its Fourier transform. To our knowledge, this method is the only one that permits to treat the problem of propagation for energies near thresholds. Hence with this method, we only prove (in the section 2.1) the

Proposition 2.1 (Propagation near thresholds). *Suppose that Assumptions 1.1 and 1.2 hold and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ be such that its support is in a sufficiently small neighborhood of $[-m;\ m]$. Then one has for $\sigma > 5/2$*

$$\|e^{-\imath tH}\mathbf{P}_c\,(H)\,\chi\,(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C\,\langle t\rangle^{-3/2}\,.$$

We recall that $\mathbf{P}_c(H)$ is defined by (1.1).
We also need to treat the propagation estimates for initial state whose spectrum does not contain any threshold. We cannot use the spectral density. So we work directly with the propagator. This is exactly the method used by Hunziker, Sigal and Soffer in [HSS99]. But in our case, their result needs some adaptation. Hence we need to generalize [HSS99, Theorem 1.1] to the case of unbounded energy. In Section 2.2, we prove the

Proposition 2.2 (Propagation far from thresholds). *Suppose that Assumption 1.1 holds. Then for any $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ bounded with support in $\mathbb{R}\backslash(-m; m)$ and for any $\sigma \geq 0$, there is $C > 0$ such that*

$$\|e^{-\imath tH}\chi\,(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C\,\langle t\rangle^{-\sigma}\,.$$

The proof of Theorem 1.1 is then a consequence of the above propositions

Proof (Proof of Theorem 1.1).
We choose $\chi_0 \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ satisfying the assumptions of Proposition 2.1, $\chi_\infty \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ satisfying assumptions of Proposition 2.2 such that $\chi_0 + \chi_\infty = 1$. Hence the continuous spectrum of H is divided in two parts. We obtain the inequality

$$\begin{aligned}\|e^{-\imath tH}\mathbf{P}_c\,(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq \|e^{-\imath tH}\chi_0\,(H)\,\mathbf{P}_c\,(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})}& \\ + \|e^{-\imath tH}\chi_\infty\,(H)\,\|_{B(L^2_\sigma, L^2_{-\sigma})}&.\end{aligned}$$

Hence from Proposition 2.1, and 2.2, we deduce Theorem 1.1.

It therefore remains to prove Propositions 2.1, and 2.2.

2.1. Step 1 : Propagation near thresholds.

2.1.1. Proof of Proposition 2.1. We now prove Proposition 2.1. Let χ be in $\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{C}^4)$, then the operator $e^{-\imath tH}\mathbf{P}_c(H)\chi(H)$ as a function of t is the Fourier transform with respect to λ of

$$\lambda \mapsto \Im R_V^+(\lambda)\mathbb{1}_{(-\infty,-m]\cup[m,\infty)}(\lambda)\chi(\lambda),$$

where

$$R_V^\pm(\lambda) = \lim_{\varepsilon\to 0^+} R_V(\lambda \pm \imath\varepsilon), \tag{2.1}$$

we will prove in Section 2.2 that the limit exists in $\mathcal{B}(L^2_\sigma, L^2_{-\sigma})$. So Proposition 2.1 is a consequence of the

Proposition 2.3. *Suppose that Assumptions 1.1 and 1.2 hold. Then for $\lambda > m$ close enough to m, one has*

$$R_V^\pm(\lambda) = \lim_{\varepsilon\to 0^+} R_V(\lambda \pm \imath\varepsilon)$$

exists in $\mathcal{B}(\mathcal{H}_\sigma^{-1/2}, \mathcal{H}_{-\sigma}^{1/2})$ for $\sigma > 3/2$. It is $\mathcal{C}^l$ if $\sigma > 1/2 + l$ and $0 < l \leq 2$ with

$$\frac{d^l}{d\lambda^l}\Im R_V^\pm(\lambda) = O(\sqrt{\lambda - m}^{\,1/2-l}), \tag{2.2}$$

as $\lambda \to m^+$.
The same holds for $\lambda < -m$ if m is replaced by $-m$.

We prove it in Section 2.1.2. The idea is then to apply to

$$\lambda \mapsto \Im R_V^+(\lambda)\mathbb{1}_{(-\infty,-m]\cup[m,\infty)}(\lambda)\chi(\lambda). \tag{2.3}$$

with $k = 1$ and $\theta = 1/2$, the following

Lemma 2.1 (Lemma 10.2 of [JK79]). *Suppose $F(\lambda) = 0$ for $\lambda > a > 0$, $F^{(k+1)} \in L^1([\delta, +\infty[)$ for any $\delta > 0$ and an integer $k \geq 0$ and that $F^{(k+1)}(\lambda) = O(\lambda^{\theta-2})$ near 0 for some $\theta \in (0,1)$. Assume further that $F^{(j)}(0) = 0$ for $j \leq k-1$, then*

$$\widehat{F}(t) = O(t^{-k-\theta}).$$

The symbol O may be replaced by o throughout.

We refer to [JK79] for the proof of Lemma 2.1. In fact to apply this lemma to (2.3), one should split this function in two parts, one supported in $\mathbb{R}^+$ and the other in $\mathbb{R}^-$. Then one translates the first one by $-m$ and applies the lemma. To deal with the other part, one works exactly in the same way after a symmetry with respect to the origin. To end the proof of Proposition 2.1, it remains to prove Proposition 2.3. This the goal of the next section.

2.1.2. Behavior near thresholds of the Dirac resolvent: proof of Proposition 2.3. In this section, our aim is to prove Proposition 2.3. First of all, we notice that if the limits (2.1) exist then we have

$$R_V^-(\lambda)^* = R_V^+(\lambda),$$

and since

$$\alpha_5(D_m + V - z)^{-1}\alpha_5 = -(D_m + \alpha_5 V \alpha_5 + z)^{-1},$$

for $\alpha_5 = \prod_{i=1}^{3} \alpha_i \beta$, one obtains

$$\alpha_5 R_V^{\pm}(\lambda)^{-1}\alpha_5 = -R^{\mp}_{\alpha_5 V \alpha_5}(-\lambda).$$

So we only need to study the behavior of $R_V^+(\lambda)$ near $+m$. Let us introduce

$$\mathbb{C}_{++} = \{z \in \mathbb{C},\, \Im z > 0,\, \Re z > 0\}$$

then the behavior for the free case ($V = 0$) is given by the

Proposition 2.4 (Dirac's resolvent expansion). *Let be $s, s' > 1/2$ with $s + s' > 2$ and $t \in \mathbb{R}$. Then $R_0(z) \in \mathcal{B}(H_s^{t-1}, H_{-s'}^t)$ is uniformly continuous in $\mathbb{C}_{++}$ and so it can be continuously extended to $\overline{\mathbb{C}_{++}}$.*
Moreover, the formal series $z \in \mathbb{C}_{++}$,

$$R_0(z) = \sum_{j=0}^{\infty} (\imath\sqrt{z^2 - m^2})^j D_m G_j + \sum_{j=0}^{\infty} z(\imath\sqrt{z^2 - m^2})^j G_j$$

with $\Im(\sqrt{z^2 - m^2}) > 0$, is an asymptotic expansion for $z \to m$ in the following sense:
Let $k \in \mathbb{N}$, if $R_0(z)$ is approximated by the corespondent finite series up to $j = k$, the remainder is $o(|z - m|^{k/2})$, as $z \to m$, in the norm of $\mathcal{B}\left(H_s^{t-1}, H_{-s'}^t\right)$ with s, $s' > k + 1/2$ (and $s + s' > 2$ if $k = 0$) and $t \in \mathbb{R}$.
In the same sense, this identity can be differentiated in z any number of times. More precisely, for $l \in \mathbb{N}^*$ the l^{th} derivative in z of the said finite series is equal to $\frac{d^l}{dz^l} R(z)$ up to an error $o(|z-m|^{k/2-l})$, as $z \to m$, in the norm of $\mathcal{B}\left(H_s^{t-1}, H_{-s'}^t\right)$ with s, $s' > k + l + 1/2$ and $t \in \mathbb{R}$.

Proof. It is an adaptation of lemmas of [JK79]. We rewrite [JK79, Lemma 2.1], [JK79, Lemma 2.2] and [JK79, Lemma 2.3] in the Dirac case with help of the identity

$$(D_m - z)^{-1}(D_m + z)^{-1} = (-\Delta + m^2 - z^2)^{-1},$$

or in $\left(\mathbb{C}^2\right)^2$

$$(D_m - z)^{-1} = \begin{pmatrix} \dfrac{z+m}{-\Delta - z^2 + m^2} & \dfrac{\sigma \cdot \nabla}{-\Delta - z^2 + m^2} \\ \dfrac{\sigma \cdot \nabla}{-\Delta - z^2 + m^2} & \dfrac{z - m}{-\Delta - z^2 + m^2} \end{pmatrix}$$

where σ are the two dimensional Pauli matrices.

To obtain the behavior of the Dirac resolvent in the general case, we would like to use the formula
$$R_V(z) = M(z)^{-1}R_0(z) \tag{2.4}$$
with
$$M(z) = (1 + R_0(z)V)\,.$$
To give a meaning to Identity (2.4), we have to prove that $M(z)$ is invertible in $\mathcal{B}(H^{1/2}_{-\sigma})$ for $\sigma > 1/2$ with $\sigma+1/2 < \rho$, where ρ is introduced in assumption 1.1. We will also give the asymptotic behavior of $R^+_V(z)$ and some of its derivatives as $\lambda \to m^+$. By means of Proposition 2.4, one has
$$z \in \overline{\mathbb{C}_{++}} \mapsto M(z) \in \mathcal{B}\left(H^{1/2}_{-\sigma}\right)$$
is uniformly continuous for $1/2 < \sigma$ and $2 < \sigma + \sigma' \le \rho$ and some $\sigma' > 1/2$. We also have
$$M(z) = M(m) + A(z),$$
with $A(z)$ uniformly continuous in $\mathcal{B}\left(H^{1/2}_{-\sigma'}, H^{1/2}_{-\sigma}\right)$ near m in $\mathbb{C}_{++}$ and tending to 0 as $\lambda \to m$ for $1/2 < \sigma$ and $2 < \sigma + \sigma' \le \rho$ and some $\sigma' > 1/2$. We now prove the

Lemma 2.2 (Threshold's eigenvector and resonance). *Suppose that Assumption 1.1 holds. Let $\mathcal{M}(s)$ be the kernel of $M(m)$ in $H^{1/2}_{-s}$ and $\mathcal{K}(s)$ the kernel of $(H-m)$ in $H^{1/2}_{-s}$. Then $\mathcal{M}(s)$ and $\mathcal{K}(s)$ are finite dimensional and do not depend on $s \in (1/2, \rho - 1/2)$. So we write $\mathcal{M}$ and $\mathcal{K}$ and we have*
$$\mathcal{M} = \mathcal{K}$$

Proof. See also [JK79, Lemma 3.1].
Let $u \in \mathcal{K}(s)$, then $(D_m + V - m)u = 0$ and $u \in H^{1/2}_{-s}$, so $Vu \in H^{-1/2}_{\rho-s}$ and since $\rho - s > 1/2$, $s > 1/2$, and $s + \rho - s > 2$, we obtain, by Proposition 2.4, $(D_m - m)^{-1}(D_m - m)u = (D_m - m)^{-1} Vu \in H^{1/2}_{-s}$. For any $\phi \in \mathcal{C}^\infty_0$,
$$\langle \phi, (D_m - m)^{-1}(D_m - m)u\rangle = \langle (D_m - m)(D_m - m)^{-1}\phi, u\rangle = \langle \phi, u\rangle,$$
we obtain $(D_m - m)(D_m - m)^{-1}Vu = Vu$ and $(D_m - m)(u + (D_m - m)^{-1}Vu) = 0$. Since $D_m - m$ has no kernel in $H^{1/2}_{-s}$, because there's no harmonic function in L^2_{-s}, we obtain $u + (D_m - m)^{-1}Vu = 0$. Hence, we have $\mathcal{K}(s) \subset \mathcal{M}(s)$.
Conversely, $I + (D_m - m)^{-1}V$ defines a Fredholm operator of $\mathcal{B}(H^{1/2}_{-s})$. If $u \in \mathcal{M}(s)$ then $u \in H^{1/2}_{-s}$ and $(D_m - m)^{-1}Vu \in H^{1/2}_{-s}$. So we write $0 = (D_m - m)(u + (D_m - m)^{-1}Vu) = (D_m - m + V)u$ and we obtain $\mathcal{M}(s) \subset \mathcal{M}(s)$.
Now we introduce $I + V(D_m - m)^{-1} \in \mathcal{B}(H^{-1/2}_s)$, and its kernel $\mathcal{N}(s)$ which is finite dimensional is a Fredholm operator. We have that $\mathcal{N}(s)$ is decreasing with s and $\mathcal{M}(s)$ is increasing. Since, by duality, $\dim \mathcal{M}(s) = \dim \mathcal{N}(s)$, we deduce that $\mathcal{N}(s)$ and $\mathcal{K}(s) = \mathcal{M}(s)$ do not depend on s.

We are now able to conclude the proof of Proposition 2.3.

Proof (Proof of Proposition 2.3). Assumption 1.2 gives $\mathcal{K} = 0$ and so with Lemma 2.2, one obtains $\mathcal{M} = 0$. Hence $M(m)$ is invertible since it is a Fredholm operator. We use Von Neumann series to obtain that $M(z)$ is invertible in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ for $\sigma > 1/2$, $2 < \sigma + \sigma' \leq \rho$ and some $\sigma' > 1/2$ and

$$M(z)^{-1} = M(m)^{-1} \sum_{j \geq 0} (A(z) M(m)^{-1})^j.$$

So for $\lambda \geq m$ close enough to m,

$$M^+(\lambda)^{-1} = \lim_{\varepsilon \to 0^+} M(\lambda + \imath\varepsilon)^{-1}$$

exists in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ for $\sigma > 1/2$ with $2 < \sigma + \sigma' \leq \rho$ and some $\sigma' > 1/2$. We obtain that $\lim_{\varepsilon \to 0^+} R_V(\lambda + i\varepsilon)$ exists in $\mathcal{B}\left(H^{-1/2}_{\sigma''}, H^{1/2}_{-\sigma}\right)$ for $\sigma > 1/2$ and $\sigma \geq \sigma'' > 1/2$ with $\sigma + \sigma'' > 2$, $2 < \sigma + \sigma' \leq \rho$ and some $\sigma' > 1/2$.
Using Proposition 2.4, we prove that if $1/2 + k < \sigma$, and $\sigma' + 1/2 + k < \rho$ then

$$\frac{d^k}{d\lambda^k} M^+(\lambda) = O(\sqrt{\lambda - m}^{1/2-k})$$

in $\mathcal{B}\left(H^{-1/2}_{-\sigma'}, H^{1/2}_{-\sigma}\right)$ for $k \in \mathbb{N}^*$ as $\lambda \to m^+$. Since we have

$$\frac{d}{d\lambda} F(\lambda)^{-1} = -F(\lambda)^{-1} \left(\frac{d}{d\lambda} F(\lambda)\right) F(\lambda)^{-1},$$

for matrix valued differentiable function F with invertible values, we obtain for $k \in \mathbb{N}^*$ the estimate

$$\frac{d^k}{d\lambda^k} M^+(\lambda)^{-1} = O(\sqrt{\lambda - m}^{1/2-k}),$$

in $\mathcal{B}\left(H^{1/2}_{-\sigma}\right)$ with $1/2 + k < \sigma$ and $\sigma + 1/2 + k < \rho$ as $\lambda \to m^+$. So by Leibniz formula, we also have for $k \in \mathbb{N}^*$

$$\frac{d^k}{d\lambda^k} R^+_V(\lambda) = O(\sqrt{\lambda - m}^{1/2-k}),$$

in $\mathcal{B}\left(H^{-1/2}_{\sigma'}, H^{1/2}_{-\sigma}\right)$ with $1/2 + j < \sigma$, $1/2 + k - j < \sigma'$ and $1/2 + k - j + \sigma < \rho$ for all $j \in \{0, \dots, k\}$ as $\lambda \to m^+$. For the case $k = 0$, we have the formula

$$R_V(z) = R_0(z) \left(1 + V R_0(z)\right)^{-1}.$$

Since $R^+(m) = R^-(m)$, this leads to

$$\Im R^+_V(m) = 0,$$

and so

$$\Im R^{\pm}_V(\lambda) = O(\sqrt{\lambda - m}^{1/2}),$$

as $\lambda \to m^+$ in $\mathcal{B}\left(H^{-1/2}_{\sigma}, H^{1/2}_{-\sigma}\right)$ with $3/2 < \sigma$ and $\sigma + 3/2 < \rho$. Hence (2.2) is proved.

2.2. Step 2: Propagation far from thresholds. In this section, we prove Proposition 2.2. We prove the propositions for $t \geq 0$. Then using $\left(e^{-\imath tH}\right)^* = e^{\imath tH}$, the result easily follows for $t \leq 0$.

2.2.1. Proof of Proposition 2.2. Let us introduce

$$A = \frac{1}{2}\left\{D_m^{-1} P \cdot Q + Q \cdot P D_m^{-1}\right\}.$$

[IM99, Lemma 3.1] gives that A is an essentially self-adjoint operator and the domain of its closure contains the domain of $\langle Q\rangle$. Proposition 2.2 is then a consequence of the

Theorem 2.1 (Minimal escape velocity). *Suppose that Assumption 1.1 holds. Then for any $\chi \in \mathcal{C}_0^\infty$ bounded with support in $(-\infty, -m) \cup (m, +\infty)$, there exists $\theta > 0$ such that for any $l \in \mathbb{R}$, for any $v \in (0, \theta)$, and any $a \in \mathbb{R}$ one has*

$$\forall t > 0,\ \left\|\mathbb{1}_{A-a-vt\leq 0} e^{-itH}\chi(H)\mathbb{1}_{A-a\geq 0}\right\| \leq Ct^{-l},$$

where C do not depend a and t.

The proof will be given in Section 2.2.2. Let us now show that Theorem 2.1 implies Proposition 2.2.

Proof (Proof of Proposition 2.2). We notice that for $c \geq 0$

$$\langle A\rangle^{-\alpha} = \langle A\rangle^{-\alpha}\mathbb{1}_{\pm A\geq ct} + O(t^{-\alpha}),$$

when $t \geq 0$, this leads to

$$\langle A\rangle^{-\alpha} e^{-itH}\chi(H)\langle A\rangle^{-\alpha} = \langle A\rangle^{-\alpha}\mathbb{1}_{A\leq \frac{(\theta-\varepsilon)t}{2}} e^{-itH}\chi(H)\mathbb{1}_{A\geq \frac{\theta t}{2}}\langle A\rangle^{-\alpha} + O(t^{-\alpha}),$$

So if we choose $a = -\frac{\theta t}{2}$ and $v = \theta - \frac{\varepsilon}{2}$ in Theorem 2.1, we obtain

$$\|\langle A\rangle^{-\alpha} e^{-itH}\chi(H)\langle A\rangle^{-\alpha}\| \leq Ct^{-\min(\alpha,\, l)}.$$

Then we prove that $\langle A\rangle^{\alpha}\langle Q\rangle^{-\alpha}$ is bounded for any positive α. It is quite immediate for integer α using multi-commutator expansion [HS00, Identity (B.24)]. To prove it for any positive real, we use [SS98, Identity (1.2)]. This identity states that for a self adjoint with $B \geq 1$ and a positive real β, we have on domain of $B^{[\alpha]+1}$

$$B^{\beta} = \frac{\sin(\pi\{\beta\})}{\pi}\int_0^{+\infty} \frac{w^{\{\beta\}-1}}{B+w} dw B^{[\beta]+1},$$

where $\{\beta\} = \beta - [\beta]$ and $[\beta]$ is the integer part. With this formula for $B = \langle A\rangle^{2k}$ for any $k \in \mathbb{N}$, we prove for any $\beta \in]0,\, 1[$ that

$$\langle A\rangle^{2k\beta} \leq C\langle Q\rangle^{2k\beta}.$$

This ends the proof of Proposition 2.2.

2.2.2. Proof of Theorem 2.1. Our proof of Theorem 2.1 is an adaptation of the one of [HSS99], we make some modifications.
For any self-adjoint operator B with domain $D(B)$, we write $Ad_A(B)$ for the operator $[A,B]$ with domain $D(A)\cap D(B)$ dense in $D(B)$, defined by

$$\forall u,v\in D(A)\cap D(B),\ \langle \imath[A,B]u,v\rangle = \imath(\langle Bu,Av\rangle - \langle Au,Bv\rangle).$$

First of all, we have

Lemma 2.3. *Suppose that Assumption 1.1 holds. Then $Ad^k_A(H)$ is bounded and can be written as a finite sum of terms of the form*

$$f(P)g(Q)h(P)$$

where f and h are rational fractions with coefficients in $\mathcal{M}_4(\mathbb{C})$ of degree at most 0 with no poles, and g is a function that satisfies Assumption 1.1.

Proof. The proof is a simple calculation based on the fact that $Ad_{P_j}(f(Q)) = -\imath(\nabla_j f)(Q)$.

We can state the

Lemma 2.4 (Mourre estimate). *If V satisfies Assumption 1.1, then for any $\theta\in(0,1)$ there exists $\nu\geq 0$, one has*

$$\mathbb{1}_{|H|\geq m+\nu}\imath[H,A]\mathbb{1}_{|H|\geq m+\nu}\geq \theta\mathbb{1}_{|H|\geq m+\nu},$$

and for any $\lambda\in(-\infty,-m)\cup(+m,+\infty)$ for any $\delta>0$ there exists $\varepsilon>0$ such that

$$\mathbb{1}_{|H-\lambda|\leq\varepsilon}\imath[H,A]\mathbb{1}_{|H-\lambda|\leq\varepsilon}\geq\left(\frac{\lambda^2}{\sqrt{\lambda^2+m^2}}-\delta\right)\mathbb{1}_{|H-\lambda|\leq\varepsilon}.$$

Proof. This is a consequence of $\imath[D_m,A]=\frac{-\Delta}{-\Delta+m^2}$ and $[V,A]$ is a compact operator in $B(L^2(\mathbb{R}^3,\mathbb{C}^4))$.

We now adapt [HSS99, Theorem 1.1] to the case of unbounded energy since here the multi-commutators $Ad^k_A(H)$ are bounded operators. We need the We introduce the

Definition 2.1. *We call generalized indicator function of $\mathbb{R}^-$ a function of the form*

$$x\mapsto e^{\frac{u(x)}{x}}\mathbb{1}_{\mathbb{R}^-}(x)$$

with $u\in\mathcal{C}^\infty_0(\mathbb{R})$ supported in $[-\eta,\eta]$ (for some $\eta>0$), nonnegative, and such that $u(0)=1$.

Note that our generalized indicator function of $\mathbb{R}^-$ are of infinite order in the sense of [HSS99, Section 2]. Using the commutators expansion presented in [HS00, Section B] and the Mourre estimate of Lemma 2.4, we have the

Lemma 2.5. *Let f be a generalized indicator function of $\mathbb{R}^-$ and $g \in \mathcal{C}^\infty(\mathbb{R})$ with support sufficiently far from thresholds $\pm m$ or support sufficiently small in $(-\infty, -m) \cup (+m, +\infty)$. Let be $A_s = s^{-1}\{A - a\}$ and $0 < \varepsilon \leq 1$. Then for any $n \in \mathbb{N}$ and $\delta > 0$ there exists a $C > 0$ independent of $a \in \mathbb{R}$ such that for $s \geq 1$*

$$g(H)i[H, f(A_s)]g(H) \leq s^{-1}\theta g(H)f'(A_s)g(H) \\ + Cs^{-1-\varepsilon}g(H)f^{1-\delta}(A_s)g(H) + Cs^{-(2n-1-\varepsilon)}g^2(H).$$

Proof. See [HSS99, Lemma 2.1], in our case we don't need to replace H by $b(H)H$ with $b \in \mathcal{C}_0^\infty$. Indeed, our commutators $Ad_A^k(H)$ are bounded by means of Lemma 2.3. Then we replace the notion of function of order p by the one of generalized indicator function. Finally, we use the fact that a generalized indicator function f satisfies

$$\forall k \in \mathbb{N}, \forall \delta \in (0,1), \exists C > 0, \quad \left|f^{(k)}\right| \leq C\left|f\right|^{1-\delta}.$$

We are now able to give the

Proof (Proof of Theorem 2.1). We write χ as a finite sum of function $g_j \in \mathcal{C}^\infty(\mathbb{R})$ with support sufficiently far from thresholds $\pm m$ or support sufficiently small in $(-\infty, -m) \cup (+m, +\infty)$. If we prove the theorem for g_j instead of χ the theorem follows by summing each estimates for g_j since the sum is finite. In the rest of the proof, we will not write the index j of g.

We notice that if $0 < v < \theta - \eta$ and if F is a positive non increasing $\mathcal{C}^\infty$-function which equals 0 on $\mathbb{R}^+$ and 1 on $(-\infty, -\eta)$, we have

$$\mathbb{1}_{(A-a-vs)<0} \leq F\left(\frac{A-a}{s} - \theta\right).$$

Now suppose F is a generalized indicator function of R^-. We consider

$$F(s^{-1}\{A - a - \theta t\})$$

and study the time evolution of the observable $g(H)f(A_{ts})g(H)$, where $f = F^2$, with respect to $e^{-\imath tH}\mathbb{1}_{A-a>0}$. That is to say we study

$$\langle e^{-\imath tH}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\imath tH}\mathbb{1}_{A-a>0}\psi\rangle.$$

We work exactly as in the proof of [HSS99, Theorem 1.1]. Hence using Lemma 2.5 we obtain for $0 \leq t \leq s$ and $s > 1$

$$\langle e^{-\imath tH}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\imath tH}\mathbb{1}_{A-a>0}\psi\rangle \leq Cs^{-(2n-2-\varepsilon)}\|\psi\|^2 \\ + Cs^{-1-\varepsilon}\int_0^t \langle e^{-\imath tH}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\imath tH}\mathbb{1}_{A-a>0}\psi\rangle^{1-\delta}||\psi||^{2\delta}.$$

Then using the Gronwall's lemma (see [ABdMG96, Lemma 7.A.1]), we obtain

$$\langle e^{-\imath tH}\mathbb{1}_{A-a>0}\psi, g(H)f(A_{ts})g(H)e^{-\imath tD_m}\mathbb{1}_{A-a>0}\psi\rangle \\ \leq \left\{Cs^{-\delta(2n-2-\varepsilon)}\|\psi\|^{2\delta} + \delta Cs^{-\varepsilon}\|\psi\|^{2\delta}\right\}^{1/\delta},$$

so if we choose a small δ and a big n, the proof is done if we choose $s = \max\{1, t\}$.

3. Proof of Theorem 1.2: dispersive estimates

Dispersive estimates for Schrödinger operators with electric potentials take place in Lebesgue spaces. This fact permits to use simple perturbation methods (like Duhamel's formula) to prove the decay estimates for perturbed Schrödinger equations. Unfortunately, we have only been able to prove dispersive estimates for Dirac operators in Besov spaces, so it was not possible for us to use Duhamel's formula or other perturbation method used for Schrödinger operators.

We notice that in the case of a Dirac operator with scalar potentials (matrix valued functions colinear with β), the square of the Dirac equation gives four coupled Klein-Gordon equations with an electrostatic potential. This permits to use results on the Klein-Gordon equation. For example, Yajima [Yaj95] proved dispersive estimates for the Klein-Gordon equation by using wave operators associated with Schrödinger operators including an electrostatic potential. But in the general case, by taking the square of a Dirac operator with a potential, we obtain also a magnetic potential. Hence the method used by Yajima does not work in our case.

To our knowledge the only one study of the dispersive estimates associated with the Dirac equation, is the the work of D'Anconna and Fanelli [DF] for the massless case. For non zero mass we have not been able to found any reference. Even for the free case for which dispersive estimates can be deduced from those of Klein-Gordon equation. Here, to give a sketch of the proof for the general case, we first prove the free case estimates (see Section 3.2), using estimates on oscillatory integrals of Section 3.1. In Section 3.3, following Cuccagna and Schirmer [CS01], we introduce the distorted plane waves. This permits us to tackle the proof of the general case in Section 3.3.2.

3.1. Estimates on some oscillatory integrals. Here, we state some stationary phase type results which will be useful for the rest of the proof. We denote by S^2 the unit sphere of $\mathbb{R}^3$.

Lemma 3.1. *Let be $f \in \mathcal{C}^1(S^2)$ and for any $v \in S^2$ and any $k \in \mathbb{R}$ define*

$$J_v(k) = \int_{S^2} e^{\imath k\{1 - v\cdot\omega\}} f(\omega)\, d\omega.$$

Then we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle} \left\{ \sum_{|\alpha|\leq 1} \int_{S^2} \frac{|\nabla^\alpha f(\omega)|}{|\omega - v|^{|\alpha|}}\, d\omega + \sum_{|\alpha|\leq 1} \int_{S^2} \frac{|\nabla^\alpha f(\omega)|}{|\omega + v|^{|\alpha|}}\, d\omega \right\}, \tag{3.1}$$

where C does not depend on f, k or v.
If f is in $\mathcal{C}^2(S^2)$ with $f(v) = f(-v) = 0$, we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle^{3/2}} \left\{ \sum_{|\alpha|\leq 1} \int_{S^2} \frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega - v|^{|\alpha|}}\, d\omega + \sum_{|\alpha|\leq 1} \int_{S^2} \frac{\left|\nabla^{2\alpha} f(\omega)\right|}{|\omega + v|^{|\alpha|}}\, d\omega \right\}, \tag{3.2}$$

where C does not depend on f, k or v.

If f is in $\mathcal{C}^2(S^2)$ and vanishes in a neighborhood of v and $-v$, we have

$$|J_v(k)| \leq \frac{C}{\langle k\rangle^2}\left\{\frac{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha}f(\omega)\right|}{|\omega-v|^{|\alpha|}}\,d\omega}{\mathrm{dist}(\mathrm{supp}(f),v)}+\frac{\sum_{|\alpha|\leq 1}\int_{S^2}\frac{\left|\nabla^{2\alpha}f(\omega)\right|}{|\omega+v|^{|\alpha|}}\,d\omega}{\mathrm{dist}(\mathrm{supp}(f),-v)}\right\}, \qquad (3.3)$$

where C does not depend on f, k or v.

Proof. We can suppose $v = (0,0,1)$ since estimate (3.1), (3.2) and (3.3) are invariant under the action of rotations. We have

$$J_v(k) = \int_0^{2\pi}\int_0^{\pi} e^{\imath k\{1-\cos(\phi)\}} f(\theta,\phi)\sin(\phi)\,d\phi d\theta,$$

then we make an integration by parts in ϕ

$$J_v(k) = -\frac{1}{k}\int_0^{2\pi}\left[e^{\imath k\{1-\cos(\phi)\}}f(\theta,\phi)\right]_0^{\pi}d\theta + \frac{1}{k}\int_0^{2\pi}\int_0^{\pi} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi f(\theta,\phi)\,d\phi d\theta,$$

If we suppose that f vanishes in a neighborhood of v or $-v$, then we use that for any ϕ'

$$|f(\theta,\phi')| \leq \int_0^{\pi}|\partial_\phi f(\theta,\phi)|\,d\phi$$

to obtain (3.1) in this case. Otherwise with help of a smooth cut-off, we split the integral in two parts, each one has a support far from v or $-v$. Repeating the previous proof for each part, we prove the estimate (3.1) in the general case.
If moreover we have $f(v) = f(-v) = 0$ then we have for any $\alpha > 0$

$$\begin{aligned}J_v(k) = \frac{1}{k}\int_0^{2\pi}\int_0^{\alpha} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi f(\theta,\phi)\,d\phi d\theta &+ \frac{1}{k}\int_0^{2\pi}\int_{\pi-\alpha}^{\pi} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi f(\theta,\phi)\,d\phi d\theta \\ &+ \frac{1}{k}\int_0^{2\pi}\int_{\alpha}^{\pi-\alpha} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi f(\theta,\phi)\,d\phi d\theta.\end{aligned}$$

We use an integration by parts to obtain for the second term of the right hand side

$$\begin{aligned}\int_{\alpha}^{\pi-\alpha} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi f(\theta,\phi)\,d\phi &= \frac{i}{k}\left[e^{\imath k\{1-\cos(\phi)\}}\frac{\partial_\phi f(\theta,\phi)}{\sin(\phi)}\right]_{\alpha}^{\pi-\alpha} \\ &- \frac{i}{k}\int_{\alpha}^{\pi-\alpha} e^{\imath k\{1-\cos(\phi)\}}\left\{\frac{\partial_\phi^2 f(\theta,\phi)}{\sin(\phi)} - \frac{\cos(\phi)\partial_\phi f(\theta,\phi)}{\sin(\phi)^2}\right\}d\phi,\end{aligned}$$

for the other terms of the right hand side direct estimations give us

$$
\begin{aligned}
|J_v(k)| \le \frac{C\alpha}{|k|}\int_0^{2\pi} \sup_\phi |\partial_\phi f(\theta,\phi)|\, d\theta \\
+ \frac{C}{\alpha|k|^2}\int_0^{2\pi}\left\{ \sup_\phi |\partial_\phi f(\theta,\phi)|\, d\theta + \int_0^{2\pi}\int_0^{\pi} \left|\partial_\phi^2 f(\theta,\phi)\right|\, d\phi d\theta \right\},
\end{aligned}
$$

choosing $\alpha = \sqrt{|k|}^{-1}$ and working like in the proof of the estimate (3.1), we obtain estimate (3.2). The reader recognized the proof of the well known Van der Corput Lemma with modification in order to give precise estimates.
For the estimate (3.3), we first split the integral $J_v(k)$ in two hemispheres with respect to the pole v and we choose $\alpha = \mathrm{dist}(\mathrm{supp}(f), v)$ or $\alpha = \mathrm{dist}(\mathrm{supp}(f), -v)$.

We obtain first the

Proposition 3.1. *Let $h \in \mathcal{C}(\mathbb{R})$ and $g \in \mathcal{C}^2(\mathbb{R}^3)$ be such that the integrals appearing in the following estimate are finite. Then defining*

$$I(k,u) = \int_{\mathbb{R}^3} e^{\imath k\{h(|\xi|)-\xi\cdot u\}} g(\xi)\, d\xi,$$

for any $u \in \mathbb{R}^3$ and any $k \in \mathbb{R}$, we have

$$|I(k,u)| \le \frac{C}{|ku|}\max_{|\alpha|\le 1}\left\{ \int_{\mathbb{R}^3} |\xi|^{|\alpha|-1}\frac{|\nabla^\alpha g(\xi)|}{\left|\frac{u}{|u|}-\frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi, \int_{\mathbb{R}^3} |\xi|^{|\alpha|-1}\frac{|\nabla^\alpha g(\xi)|}{\left|\frac{u}{|u|}+\frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi \right\} \tag{3.4}$$

where C does not depend on h, g, k or u.
If moreover g vanishes in a cone of axis $D = \mathrm{Span}(u)$, we have

$$
\begin{aligned}
|I(k,u)| \le \frac{C}{|ku|^2\,\mathrm{dist}(\mathrm{supp}(g)\cap S^2, D\cap S^2)} \times \\
\times \max_{|\alpha|\le 1}\left\{ \int_{\mathbb{R}^3} |\xi|^{2|\alpha|-2}\frac{\left|\nabla^{2\alpha} g(\xi)\right|}{\left|\frac{u}{|u|}-\frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi, \int_{\mathbb{R}^3} |\xi|^{2|\alpha|-2}\frac{\left|\nabla^{2\alpha} g(\xi)\right|}{\left|\frac{u}{|u|}+\frac{\xi}{|\xi|}\right|^{|\alpha|}}\, d\xi \right\}
\end{aligned}
$$

where C does not depend on h, g, k or u.

Proof. We write

$$I(k,u) = \int_{\mathbb{R}^3} e^{\imath k\{h(|\xi|)-\xi\cdot u\}} g(\xi)\, d\xi = \int_{\mathbb{R}^+} e^{\imath k\{h(\rho)-\rho|u|\}} J_{\frac{u}{|u|},\rho}(\rho k|u|)\rho^2\, d\rho,$$

where $J_{v,\rho}(k) = \int_{S^2} e^{\imath k\{1-v\cdot\omega\}} g(\rho\omega)\, d\omega$ and we apply Lemma 3.1.

We introduce a first useful variant with the

Proposition 3.2. *Let $g \in \mathcal{C}^{1+k}(\mathbb{R}^3)$ be such that the integrals appearing in the following estimate are finite. We introduce*

$$F(x) = \int_{\mathbb{R}^3} e^{\imath\{|\xi||x|-\xi\cdot x\}} g(\xi)\, d\xi$$

for any $x \in \mathbb{R}^3$. Then for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| \le k$ we have

$$|\nabla^\alpha F(x)| \le \frac{C}{|x|^{|\alpha|+1}} \max_{|\beta|\le 1+|\alpha|} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\beta|-1} \frac{|\nabla^\beta g(\xi)|}{\left|\frac{x}{|x|} - \frac{\xi}{|\xi|}\right|}\, d\xi \right\}. \tag{3.5}$$

If moreover g vanishes in a half cone of axis $D^+ = \left\{\rho \frac{x}{|x|},\ \rho \in \mathbb{R}^+\right\}$. Then for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| \le k$, we have

$$|\nabla^\alpha F(x)| \le \frac{C}{|x|^{|\alpha|+2}\mathrm{dist}(\mathrm{supp}(g) \cap S^2, D^+ \cap S^2)} \times$$
$$\times \max_{|\beta|\le 2+|\alpha|} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\beta|-2} \frac{|\nabla^\beta g(\xi)|}{\left|\frac{x}{|x|} - \frac{\xi}{|\xi|}\right|}\, d\xi \right\}. \tag{3.6}$$

Proof. The critical points correspond to the the semi axis spanned by x. We treat the part of the integrals which is far from critical points by using an integration by parts with help of the operator $L = \frac{\frac{\xi}{|\xi|} - \frac{x}{|x|}}{|x|\left|\frac{\xi}{|\xi|} - \frac{x}{|x|}\right|^2} \cdot \nabla_\xi$. Let be $\phi(x,\xi) = \{|\xi||x| - \xi \cdot x\}$, we have

$$F(x) = \langle L e^{\imath\phi(x,\cdot)}, g\rangle = \langle e^{\imath\phi(\xi,\cdot)}, L^* g\rangle,$$

with

$$L^* = -L - \frac{2}{|x||\xi|\left|\frac{\xi}{|\xi|} - \frac{x}{|x|}\right|^2}.$$

This gives the bound

$$\frac{C}{|x|} \max_{|\beta|\le 1} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\beta|-1} |\nabla^\beta g(\xi)|\, d\xi \right\},$$

or after an iteration

$$\frac{C}{|x|^2} \max_{|\beta|\le 2} \left\{ \int_{\mathbb{R}^3} |\xi|^{|\beta|-1} |\nabla^\beta g(\xi)|\, d\xi \right\},$$

we obtain Estimate (3.5) for $\alpha = 0$. The method to treat the other part of the integral is exactly the one we used in the proof of Proposition 3.1.
For higher order derivatives, we have

$$\nabla_x e^{\imath k\phi(x,\xi)} = \frac{|\xi|}{|x|} \nabla_\xi e^{\imath k\phi(x,\xi)}$$

and so

$$\langle \nabla_x e^{\imath\phi(x,\cdot)}, g\rangle = -\frac{1}{|x|} \langle e^{\imath\phi(x,\cdot)}, \nabla |Q| g\rangle.$$

the result is then obtained by applying this trick $|\alpha|$ times and then repeating our proof for the case $\alpha = 0$, we obtain Estimates (3.5) and (3.6) for $\nabla^\alpha F(x)$.

And finally, we need the

Proposition 3.3. *Let be $g \in \mathcal{C}^2(\mathbb{R}^3)$ with compact support. Then for any $u \in \mathbb{R}^3$, $k \in \mathbb{R}$ and*

$$I(k,u) = \int_{\mathbb{R}^3} e^{\imath k\{\sqrt{\xi^2+m^2}-\xi\cdot u\}} g(\xi)\, d\xi,$$

we have

$$|I(k,u)| \leq \frac{C}{|k|^{3/2}} \max\left[\max_{|\alpha|\leq 2} \left\{ \int_{\mathbb{R}^3} \left| \langle\xi\rangle^{|\alpha|-1} \nabla^\alpha g(\xi) \right| \, d\xi \right\}; \right.$$
$$\left. \frac{1}{|u|\sqrt{\inf\limits_{x\in \mathrm{supp}(g)} \left\{ \frac{m^2}{\sqrt{x^2+m^2}^3} \right\}}} \max_{\substack{l\leq 1\\ n\leq 1}} \left\{ \int_{\mathbb{R}^3} |\xi|^{l-n-1} \frac{\left|\partial^l_{|\xi|} \partial^n_\omega g(\xi)\right|}{\left|\frac{u}{|u|} - \frac{\xi}{|\xi|}\right|} d\xi \right\} \right]. \qquad (3.7)$$

Proof. We can suppose $u = (0,0,|u|)$ since estimate (3.7) is invariant under the action of rotations.

The oscillatory integral $I(k,u)$ is bounded and critical points of the phase of $I(k,u)$ are supported by the semi axis spanned by u. With help of a smooth cut-off function χ , we split the integral in two parts $I(k,u) = I_1(k,u) + I_2(k,u)$, where $I_1(k,u)$ is supported in a half cone around u. We then use multiple integrations by parts with help of the operator

$$L = \frac{\frac{\xi}{\sqrt{\xi^2+m^2}} - u}{|k| \left| \frac{\xi}{\sqrt{\xi^2+m^2}} - u \right|^2} \cdot \nabla_\xi.$$

Since $(1-\chi)\, g \in \mathcal{C}^2(\mathbb{R}^3)$ has support far from critical points and since for $\lambda(\xi) = \sqrt{\xi^2+m^2}$ we have $\left\| \nabla^\alpha_\xi \lambda(\xi) \right\| \leq C_\alpha \lambda(\xi)^{1-|\alpha|}$, we obtain

$$|I_1(k,u)| \leq \frac{C}{|k|} \sum_{|\alpha|\leq 1} \|\lambda(Q)^{|\alpha|-1} \nabla^\alpha g\|_{L^1}$$

and

$$|I_1(k,u)| \leq \frac{C}{k^2} \sum_{|\alpha|\leq 2} \|\lambda(Q)^{|\alpha|-1} \nabla^\alpha g\|_{L^1}.$$

Otherwise $I_2(k,u)$ has support in a small cone around u, and we have

$$\begin{aligned} I_2(k,u) &= \int_{\mathbb{R}^3} e^{\imath k\{\sqrt{\xi^2+m^2}-\xi\cdot u\}} \widetilde{g}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^+} e^{\imath k\{\sqrt{\xi^2+m^2}-|\xi||u|\}} J_{\frac{u}{|u|},\rho}(\rho k|u|) \rho^2\, d\rho, \end{aligned}$$

with

$$J_{v,\rho}(k) = \int_{S^2} e^{\imath k\{1-v\cdot\omega\}} \widetilde{g}(\rho\omega)\, d\omega,$$

where $\widetilde{g} = \chi g$. We obtain after an integration by parts

$$J_{v,\rho}(k) = -\frac{1}{k}\int_0^{2\pi}\left[e^{\imath k\{1-\cos(\phi)\}}\widetilde{g}(\rho\omega(\theta,\phi))\right]_0^{\pi} d\theta + \frac{1}{k}\int_0^{2\pi}\int_0^{\pi} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta,$$

Since we assumed $\widetilde{g}$ is supported in half cone around u, we have $\widetilde{g}(\rho\omega(\theta,\pi)) = 0$. Hence we obtain

$$J_{v,\rho}(k) = \frac{1}{k}\int_0^{2\pi}\int_0^{\pi}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\theta + \frac{1}{k}\int_0^{2\pi}\int_0^{\pi} e^{\imath k\{1-\cos(\phi)\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta,$$

and so

$$I_2(k,u) = \frac{1}{|k|\,|u|}\int_{\mathbb{R}^+}\int_0^{2\pi}\int_{-\pi}^{\pi} e^{\imath k\{\sqrt{\rho^2+m^2}-\rho|u|\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta \rho\, d\rho + \frac{1}{|k|\,|u|}\int_{\mathbb{R}^+}\int_0^{2\pi}\int_{-\pi}^{\pi} e^{\imath\{\sqrt{\rho^2+m^2}-\rho|u|\cos(\phi)\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\, d\phi d\theta\rho\, d\rho. \quad (3.8)$$

Let us now study the decay resulting from the dispersive behavior of the radial part. To this end, we follow the proof of the well-known Van Der Corput lemma. We study

$$L(k,u,\phi,\phi',\theta) = \int_{\mathbb{R}^+} e^{\imath|k|\{\sqrt{\rho^2+m^2}-\rho|u|\cos(\phi')\}}\partial_\phi\widetilde{g}(\rho\omega(\theta,\phi))\rho\, d\rho.$$

Notice that, in view of (3.8), we are only interested by $L(k,u,\phi,\phi,\theta)$ and $L(k,u,\phi,0,\theta)$. First, for any differentiable function on $\mathbb{R}$ such that $|f'| \geq 1$, we have for any $\alpha \in \mathbb{R}^+$

$$\lambda\left(\{t \in \mathbb{R};\, |f(t)| \leq \alpha\}\right) \leq \alpha, \quad (3.9)$$

for λ the Lebesgue measure. We introduce

$$h(\rho) = \sqrt{\rho^2+m^2} - \rho|u|\cos(\phi'),$$

and we apply (3.9) to

$$f(\rho) = \frac{\partial_\rho h(\rho)}{\inf\limits_{x\in\operatorname{supp}(g)}\left\{|\partial^2_{|x|}h(|x|)|\right\}}.$$

We notice that $\partial^2_\rho h(\rho)$ does not depend on u or ϕ'. With help of a smooth cut-off function, we split the integral L in two parts, one has support

$$\left\{\rho \in \mathbb{R}^+;\, |f(\rho)| < \alpha\right\}$$

and the other is its complementary. In fact, we obtain exactly three interval corresponding to

$$\left\{\rho \in \mathbb{R}^+;\, f(\rho) < -\alpha\right\},\ \left\{\rho \in \mathbb{R}^+;\, -\alpha \le f(\rho) \le \alpha\right\},\ \left\{\rho \in \mathbb{R}^+;\, \alpha < f(\rho)\right\}.$$

In the first and third interval, we make an integration by parts and in the second interval, we use Estimate (3.9). Hence we obtain the bound

$$\begin{aligned}
&|L(k,u,\phi,\phi',\theta)| \le \\
&\quad \max\left\{\alpha,\ \frac{1}{\alpha|k| \inf\limits_{x\in \mathrm{supp}(\widetilde{g})}\left\{|\partial^2_{|x|}h(|x|)|\right\}}\right\} \max_{\rho\in\mathbb{R}^+}\left\{\rho\,|\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))|\right\} \\
&\qquad + \frac{1}{\alpha|k| \inf\limits_{x\in \mathrm{supp}(\widetilde{g})}\left\{|\partial^2_{|x|}h(|x|)|\right\}} \times \\
&\qquad \times \max\left\{\int_{\mathbb{R}^+} |\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))|\ d\rho,\ \int_{\mathbb{R}^+} \rho\,|\partial_\rho\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))|\ d\rho\right\}.
\end{aligned}$$

We use

$$\rho\,|\partial_\phi \widetilde{g}(\rho\omega(\theta,\phi))| \le 2\max\left\{\int_{\mathbb{R}^+} |\partial_\phi \widetilde{g}(r\omega(\theta,\phi))|\ dr,\ \int_{\mathbb{R}^+} r\,|\partial_\rho\partial_\phi \widetilde{g}(r\omega(\theta,\phi))|\ dr\right\}$$

and then choose

$$\alpha = \frac{1}{\sqrt{|k| \inf\limits_{x\in \mathrm{supp}(\widetilde{g})}\left\{|\partial^2_{|x|}h(|x|)|\right\}}},$$

and plugging the resulting estimates for $\phi' = 0$ and $\phi' = \phi$ in (3.8), we obtain estimate (3.7).

3.2. Dispersive estimates for the free case equation. Thanks to the tools introduced in Section 3.1, we are able to state the

Theorem 3.1 (Dispersive estimates for free Dirac operator). *For any $p \in [1,2]$, for all $\theta \in [0;1]$, for all $s, s' \in \mathbb{R}$, such that $s - s' \ge (\frac{2}{p} - 1)(2+\theta)$ and $q \in [1,\infty]$, we have*

$$\|e^{-\imath t D_m}\|_{B^s_{p,q}, B^{s'}_{p',q}} \le (K(t))^{\frac{2}{p}-1},$$

with

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in [0,1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1,\infty), \end{cases}$$

and $p' = \frac{p}{p-1}$.

Proof. The proof is a straightforward adaptation of the one of Brenner in Appendix 2 of [Bre85] for the Klein-Gordon equation. We give a sketch of the proof for the reader's convenience. Note that the proof of Theorem 1.2, for non zero potential, is based on the same ideas.
We only need to prove the case $p = 1$, since the general case follows by interpolation of the case $p = 1$ and the charge conservation which corresponds to the case $p = 2$. Then using $D_m = \sqrt{-\Delta + m^2}(\pi_+ - \pi_-)$ with $\pi_\pm = 1\!\!1_{\mathbb{R}^\pm}(D_m) = \frac{1}{2}\left\{1 \pm |D_m|^{-1} D_m\right\}$, we obtain the estimates from those relative to the relativistic Schrödinger operator $\sqrt{-\Delta + m^2}$:

$$\|e^{-\imath t\sqrt{-\Delta+m^2}}\|_{B^s_{1,q}, B^{s'}_{\infty,q}} \le K(t)$$

which in turn follow from

Proposition 3.4. *For any $\chi \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^4)$, we define $\chi_j(x) = \chi(2^{-j}|x|)$. Then for $\theta' \in [0, 1]$, we have:*

1. *if $0 \notin \operatorname{supp}(\chi)$,*

$$\|e^{-\imath t\sqrt{-\Delta+m^2}}\chi_j\left(\sqrt{-\Delta + m^2}\right)\|_{L^1,\, L^\infty} \le C2^{(2+\theta')j}|t|^{-(1+\theta'/2)} \tag{3.10}$$

where C is independent of t and j.

2. *if $0 \in \operatorname{supp}(\chi)$,*

$$\|e^{-\imath t\sqrt{-\Delta+m^2}}\chi(-\Delta + m^2)\|_{L^1,\, L^\infty} \le C\langle t\rangle^{-3/2} \tag{3.11}$$

where C is independent t.

We postpone the proof of Proposition 3.4 until the end of the proof of Theorem 3.1.

We have

$$\left\|e^{-\imath t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta + m^2})f\right\|_\infty \le C2^{3j}\|f\|_1$$

interpolating with Estimate (3.10) of Proposition 3.4 for $\theta' = 0$ when $t \le 1$ and using Estimate (3.10) with $\theta' = \theta$ for $t \ge 1$, one obtains

$$2^{js'}\left\|e^{-\imath t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta + m^2})f\right\|_\infty \le C\kappa_j(t)2^{js}\|f\|_1$$

with

$$\kappa_j(t) = 2^{j(2+\theta+s'-s)}\begin{cases} |t|^{-1+\theta/2} \text{ if } |t| \le 1, \\ |t|^{-1-\theta/2} \text{ if } |t| \ge 1. \end{cases}$$

We use

$$\sup_{j\in\mathbb{N}} \kappa_j \le CK(t)$$

if $2 + \theta \le s - s'$ and Estimate (3.11) to prove Theorem 3.1. Hence to conclude the proof, we need to give the

Proof (Proof of Proposition 3.4). Estimates of the same type, but for $\mathcal{B}\left(L^p, L^{p'}\right)$ spaces with $p \in [4/3, 2]$ can be found in [MSW80,MSW79]. In the present case $p = 1$ a proof can be found in [Bre77]. This proof, which covers a much more general situation, is quite complicated. We propose here a simpler proof inspired by [CS01]. The kernel of $e^{-\imath t\sqrt{-\Delta+m^2}}\chi_j(\sqrt{-\Delta+m^2})$ is given by $(2\pi)^{-3/2}K_j(x-y)$ where

$$K_j(x,t) = \int_{\mathbb{R}^3} e^{-\imath t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi$$

Hence, we estimate the L^∞ norm of K_j.
If $|x|/|t| \ll 2^{j-1}/\sqrt{2^{2j-2}+m^2}$ or $|x|/|t| \gg 1$, we use non stationary phase lemma in $\mathbb{R}^3$ with help of the operator $L = \frac{\frac{\xi}{\sqrt{\xi^2+m^2}}-x}{t\left|\frac{\xi}{\sqrt{\xi^2+m^2}}-x\right|^2}\cdot\nabla$. Hence, in this case, we obtain the estimate

$$\left|\int_{\mathbb{R}^3} e^{-\imath t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi\right| \le C_n 2^{-(n-3)j}|t|^{-n},$$

for any $n \in \mathbb{N}$. Otherwise, we apply Proposition 3.1 with $h(r) = \sqrt{r^2+m^2}$, $k = t$, $u = x/t$ and $g(x) = \chi_j(|x|)$. So if $0 \notin \operatorname{supp}(\chi)$, to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} e^{-\imath t\sqrt{\xi^2+m^2}+x\cdot\xi}\chi_j(\sqrt{\xi^2+m^2})\,d\xi\right|\\ &\le \frac{C}{|t|}\max_{|\beta|\le 1}\int_{\mathbb{R}^3}|\xi|^{|\beta|-1}\frac{\left|\nabla^\beta\chi_j(\sqrt{\xi^2+m^2})\right|}{\left|\frac{x}{|x|}\pm\frac{\xi}{|\xi|}\right|}d\xi\\ &\le \frac{C2^{2j}}{|t|}.\end{aligned}$$

Notice that in this case, $|x|/|t| \ge c' > 0$. If instead of Proposition 3.1, we use Proposition 3.3 with $g = \chi_j$, $k = t$ and $u = x/t$, we prove the estimate

$$|K_j(x,t)| \le \frac{C2^{3j}}{|t|^{3/2}}.$$

The estimate (3.10) is then obtained by interpolation. For (3.11), we use the classical stationary (Morse lemma) and non-stationary phase methods (integration by parts) in $\mathbb{R}^3$. For more details about the method one can look at the end of the proof of Proposition 3.9. This ends the proof of 3.4.

This ends the proof of Theorem 3.1.

3.3. Distorted Plane Waves. Our aim is now to generalize the previous method to the perturbed case. Let us introduce the wave operators

$$W^\pm = \lim_{t\to\pm\infty} e^{\imath t(D_m+V)}e^{-\imath tD_m} \tag{3.12}$$

(for the existence and the completeness : $\mathrm{Ran}(W^{\pm}) = \mathrm{Ran}(\mathbf{P}_c(H))$ of these operator see [GM01, Theorem 1.5]). With the intertwining property

$$f(H)W^{\pm} = W^{\pm}f(D_m), \tag{3.13}$$

for any bounded borelian function f, and Fourier transform $\mathcal{F}$, we shall obtain for $h(\xi) = \alpha \cdot \xi + m\beta$

$$e^{-\imath tH}\mathbf{P}_c(H) = W^{\pm}e^{-\imath tD_m}W^{\pm *} = W^{\pm}\mathcal{F}e^{-\imath th(Q)}\left(W^{\pm}\mathcal{F}\right)^* .$$

So we can adapt the previous method if we are able to prove some estimates about the kernel ψ_V of $W^{\pm}\mathcal{F}$. The kernel ψ_V is called *distorted plane wave*. We notice that ψ_V is a 4×4 matrix valued function.
We will show that the previous method works with $\psi_V\psi_V^*\chi_j$ in place of χ_j with small modifications. So we need estimates on ψ_V. Generally, distorted plane waves are studied like perturbations of free plane waves. So we will prove estimates on the perturbative part, written w in the sequel.

3.3.1. Definition and properties. We need to introduce the free plane wave. Let

$$h(k) = \alpha \cdot k + m\beta$$

for any $k \in \mathbb{R}^3$, notice that $D_m = h(P)$. This hermitian matrix has for eigenvectors the

$$\psi_0^j(k,x) = e^{\imath k\cdot x}u(k)e_j$$

where

$$u(k) = \frac{(m + \lambda(k))Id - \beta\alpha \cdot k}{\sqrt{2\lambda(k)(m + \lambda(k))}} \tag{3.14}$$

with $\lambda(k) = \sqrt{k^2 + m^2}$ and $(e_j)_j$ are vectors of the canonical basis of $\mathbb{C}^4$. For more details see [Tha92, Section 1.4, Section 1.F].
By definition, a distorted plane wave is a solution of the PDO equation

$$(D_m + V)\,\psi = \pm\sqrt{k^2 + m^2}\psi \tag{3.15}$$

with for some j and any $k \in \mathbb{R}^3$, $\psi(k,x) - \psi_0^j(k,x)$ tending to zero as x goes to infinity (in some sense), see [Agm75, section 5].
A solution of (3.15) is a function $\psi(k,x)$ of two variables here k is a 3-dimensional vector which is called the wave vector. A free plane wave ψ_0^j satisfies the PDO equation (3.15) in the case $V = 0$. Following [Agm75], we introduce two families of function

$$\psi_V^j(k,x) = \psi_0^j(k,x) - \left\{R_V^+(\lambda(k))V(\cdot)\psi_0^j(k,\cdot)\right\}(x)$$

for $j \in \{1,2\}$ and

$$\psi_V^j(k,x) = \psi_0^j(k,x) - \left\{R_V^+(-\lambda(k))V(\cdot)\psi_0^j(k,\cdot)\right\}(x)$$

for $j \in \{3,4\}$. The rest of the proof works also for R_V^- instead of R_V^+ (the trace of the resolvent $R_V^{\pm}$ was introduced in (2.1)).
In case there is no resonance at thresholds and no eigenvalue at thresholds, Theorem 1.1 gives us that $R_V^+(\lambda(p))$ is in $B(L^2_\sigma, L^2_{-\sigma})$ for any $\sigma > 5/2$, this also work if $\sigma \geq 1$ see Proposition 3.10 below. So the previous definition make sense if Assumption 1.1 holds and we have the

Proposition 3.5. *Suppose that Assumptions 1.1 and 1.2 hold. Then for any* $k \in \mathbb{R}^3 \setminus \{0\}$, $\psi_V^j(k,x)$ *satisfies equation* (3.15).

Distorted plane waves define a generalized Fourier transform. We introduce $\psi_V(k,x) \in \mathcal{M}_4(\mathbb{C})$ the matrix with vector column $\psi_V^j(k,x)$ and we define

$$(\mathcal{F}_V f)(k) = \int_{\mathbb{R}^3} \psi_V(k,x) f(x)\, dx,$$

which is *a priori* defined on the Schwartz space $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ but will be extended to L^2.

Distorted plane waves are also called generalized eigenfunctions, since they correspond to "eigenvalues" associated with the continuous spectrum. Indeed, we can prove the

Theorem 3.2 (Eigenfunction Expansion). *The operator $\mathcal{F}_V$ defines a bounded linear map from L^2 into itself. Its kernel is given by the the sum of the eigenspaces of H. Moreover it is a unitary map from $\mathbf{P}_c(H)L^2$ onto $L^2(\mathbb{R}^3)$ with*

$$(\mathcal{F}_V^* f)(x) = \lim_{n\to\infty} \int_{K_n} \psi_V(k,x)^* f(k)\, dk,$$

for any $(K_n)_{n\in\mathbb{N}}$ a family of compact sets with $K_n \subset K_{n+1}$ and $\cup_{n\in\mathbb{N}} K_n = \mathbb{R}^3$. Finally, for any interval $I \subset \mathbb{R}$, one has

$$\|\mathbb{1}_I(H) f\|^2 = \int_{\sigma(h(k))\cap I \neq \emptyset} |\mathcal{F}_V f|^2\, dk \tag{3.16}$$

where $\sigma(h(k))$ is the spectrum of $h(k)$.

Proof. The proof is an easy adaptation of the proof of [Agm75, Theorem 6.2] (see also [RS79, Theorem XI.41]), the main difference is that here we insert the unitary matrix u defined in (3.14). Formula (3.16) is nothing more than an adaptation of [Agm75, Formula (6.6)] or [RS79, Formula 82e'].

We also have the

Lemma 3.2 (Intertwining Property). *Let g be a bounded borelian function with support in $\mathbb{R} \setminus (-m, m)$, we have*

$$\mathcal{F}_V g(H) = (g \circ h) \mathcal{F}_V. \tag{3.17}$$

Proof. Using (3.16), we obtain that (3.17) is true for $g = \mathbb{1}_I$ with I an interval of $\mathbb{R} \setminus (-m, m)$. We then obtain it for bounded borelian function with support in $\mathbb{R} \setminus (-m, m)$, usual density arguments and properties of functional calculus. More precisely, we use the fact that a bounded sequence of borelian functions which converges everywhere gives a sequence of bounded operators which converge strongly.

Hence we deduce that, for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, the kernel of $e^{-\imath tH}\chi(H)$ is given by

$$\left[e^{-\imath tH}\chi(H)\mathbf{P}_c(H)\right](x,y) = \int_{\mathbb{R}^3} \psi_V(k,x)^* e^{-\imath th(k)}\chi(h(k))\psi_V(k,y)\,dk.$$

which exactly means

$$e^{-\imath tH}\chi(H)\mathbf{P}_c(H) = (\mathcal{F}_V)^* e^{-\imath th}\chi(h)\mathcal{F}_V \tag{3.18}$$

We recall that we want to prove the decay of $e^{-\imath tH}\chi(H)$ as $t \to +\infty$ in some Besov spaces. We observe that

$$e^{-\imath th(k)}\chi(h(k)) = e^{-\imath t\lambda(k)}\chi(\lambda(k))P_+(k) + e^{\imath t\lambda(k)}\chi(-\lambda(k))P_-(k),$$

where $P_+(k)$ (resp. $P_-(k)$) is the projector associated with the positive (resp. negative) part of the spectrum of $h(k)$, *i.e.*

$$P_\pm(k) = \frac{1}{2}\left(1 \pm \frac{h(k)}{\lambda(k)}\right).$$

Hence, in the following we study the functions

$$(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \int_{\mathbb{R}^3} e^{\mp\imath t\lambda(k)}\left(P_\pm\psi_V(k,x)\right)^*\left(P_\pm\psi_V(k,y)\right)\chi(h(k))\,dk.$$

3.3.2. End of the proof of Theorem 1.2. We now prove Theorem 1.2 with help of three propositions which will be proven in Section 3.3.3. These propositions give some estimates on the perturbed part of the distorted plane wave. Following Cuccagna and Schirmer in [CS01], we write $\psi_V(k,x) = e^{\imath k\cdot x}(u(k) + w(k,x))$ where w is the perturbation part which satisfies

$$w(k,x)_j = \begin{cases} e^{-\imath k\cdot x}\left\{R_V^+(+\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if} \quad j \in \{1,\,2\}, \\ e^{-\imath k\cdot x}\left\{R_V^+(-\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if} \quad j \in \{3,\,4\}, \end{cases} \tag{3.19}$$

and we now state our propositions.

Proposition 3.6. *Suppose that Assumptions 1.1 and 1.2 hold. Then there exists $C > 0$ such that for any $k,\, x \in \mathbb{R}^3 \setminus \{0\}$, and any $\beta \in \mathbb{N}^3$ with $|\beta| \le 1$, one has*

$$\left|\nabla_k^\beta w(k,x)\right| \le \frac{C}{\langle k\rangle^{|\beta|}} \frac{\langle x\rangle^{|\beta|}}{\left\langle |x|\left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle}. \tag{3.20}$$

Moreover one has

$$|\nabla_k w(k,x)| \le C\frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\}\left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle^2}. \tag{3.21}$$

We use this to prove the time decay in $|t|^{-1}$. Unfortunately this doesn't work for the $|t|^{-3/2}$ decay, hence we then study

$$v(k,x)_j = \begin{cases} e^{\imath|k||x|}\left\{R_V^{+}(+\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j\in\{1,\,2\},\\ e^{-\imath|k||x|}\left\{R_V^{+}(-\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j\in\{3,\,4\},\end{cases} \tag{3.22}$$

and

$$\widetilde{v}(k,x)_j = \begin{cases} e^{\imath|k||x|}\left\{\nabla_k R_V^{+}(+\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j\in\{1,\,2\},\\ e^{-\imath|k||x|}\left\{\nabla_k R_V^{+}(-\lambda(k))\left\{Ve^{\imath k\cdot Q}u(k)_j\right\}\right\}(x), & \text{if } j\in\{3,\,4\}.\end{cases} \tag{3.23}$$

One has the

Proposition 3.7. *Suppose that Assumptions 1.1 and 1.2 hold. Then if $\rho > 3 + |\beta|$ for some $\beta\in\mathbb{N}^3$, there exists $C>0$ such that for any $k,\, x\in\mathbb{R}^3\setminus\{0\}$, one has*

$$|\nabla_k^{\beta} v(k,x)| \le \frac{C}{\left\langle |x|\left|\frac{x}{|x|}-\frac{k}{|k|}\right|\right\rangle}.$$

Proposition 3.8. *Suppose that Assumptions 1.1 and 1.2 hold. Then if $\rho > 3 + |\beta|$ for some $\beta\in\mathbb{N}^3$, there exists $C>0$ such that for any $k,\, x\in\mathbb{R}^3\setminus\{0\}$, one has*

$$|\nabla_k^{\beta} \widetilde{v}(k,x)| \le \frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\}\left|\frac{x}{|x|}-\frac{k}{|k|}\right|\right\rangle^{2}}.$$

Using Proposition 3.6, 3.7 and 3.8 (which are proved in Section 3.3.3 below), let us prove the following

Proposition 3.9. *Suppose that Assumptions 1.1 and 1.2 hold. Then we have for $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ with support in $\mathbb{R}\setminus[-m;m]$ for any $\theta\in[0,1]$ and $j\in\mathbb{N}$,*

$$\left\|e^{-\imath tH}\chi(2^{-j}H)\right\|_{L^1\to L^\infty} \le \frac{C2^{(2+\theta)j}}{|t|^{1+\theta/2}}, \tag{3.24}$$

with C independent of t and j.
We also have for $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$, for any $\theta\in[0,1]$

$$\left\|e^{-\imath tH}\chi(H)\mathbf{P}_c(H)\right\|_{L^1\to L^\infty} \le \frac{C}{|t|^{1+\theta/2}}, \tag{3.25}$$

with C independent of t.

Proof. The proof works like the one of Proposition 3.4 with some modifications due to the fact that high derivatives in k of $w(k,x)$ grow with respect to x.

We need the L^∞ norm of the kernel of $e^{-\imath tH}\chi(2^{-j}H)$. This kernel thanks to (3.18) is given by

$$\begin{aligned} I_j(t,x,y) = &\int_{\mathbb{R}^3} e^{-\imath t\sqrt{\xi^2+m^2}} e^{-i\xi\cdot(x-y)} \left\{P_+(\xi)(u^*(\xi)+w^*(x,\xi))\right\} \times \\ &\qquad \times \left\{P_+(\xi)(u(\xi)+w(y,\xi)\right\} \chi(2^{-j}\lambda(\xi))\, d\xi \\ &+ \int_{\mathbb{R}^3} e^{+\imath t\sqrt{\xi^2+m^2}} e^{-i\xi\cdot(x-y)} \left\{P_-(\xi)(u^*(\xi)+w^*(x,\xi))\right\} \times \\ &\qquad\qquad \times \left\{P_-(\xi)(u(\xi)+w(y,\xi)\right\} \chi(-2^{-j}\lambda(\xi))\, d\xi. \end{aligned}$$

We notice that if we expand each integrand in terms of u and w, we obtain the sum of the integrals

$$\begin{aligned} &I_j^{\pm}[z,z'](t,x,y) \\ &\quad = \int_{\mathbb{R}^3} e^{\mp\imath t\sqrt{\xi^2+m^2}} e^{-\imath\xi\cdot(x-y)} \left\{P_\pm(\xi) z^*(x,\xi) P_\pm(\xi) z'(y,\xi)\right\} \chi(\pm 2^{-j}\lambda(\xi))\, d\xi. \end{aligned}$$

with $z, z' \in \{u, w\}$. We notice that $I_j^+[u,u](t,x,y) + I_j^-[u,u](t,x,y)$ is the kernel of $e^{-itD_m}\chi_j(D_m)$, hence we only treat the other integrals.

For the $|t|^{-1}$ decay, if $|x-y|/|t| \ll 2^{j-1}/\sqrt{2^{2j-2}+m^2}$ or $|x-y|/|t| \gg 1$, the phase has no critical point. We use an integration by parts in $\mathbb{R}^3$ with help of the operator

$$L = \frac{\left(\frac{\xi}{\sqrt{\xi^2+m^2}} - x\right)}{t\left|\frac{\xi}{\sqrt{\xi^2+m^2}} - x\right|^2} \cdot \nabla_\xi.$$

So with the estimate (3.20) of Proposition 3.6 and with

$$\left|\partial_i \frac{\xi}{\sqrt{\xi^2+m^2}}\right| \le \frac{C}{|\xi|},$$

we obtain the estimate

$$\left|I_j^{\pm}[z,z'](t,x,y)\right| \le C 2^{2j}|t|^{-1},$$

with C independent of j and t.

Otherwise if $|x-y|/|t| \ge c > 0$, using first (3.4) of Proposition 3.1 and then (3.21) of Proposition 3.6, we infer

$$\left|I_j^{\pm}[z,z'](t,x,y)\right| \le C 2^{2j}|t|^{-1},$$

with C independent of j and t.

For the $|t|^{-3/2}$ decay, first if $|x-y|/|t| \ge c > 0$, we write

$$I_j^{\pm}[z,z'](t,x,y) = \int_{\mathbb{R}^+} e^{\mp\imath t\sqrt{\rho^2+m^2} - \imath\rho|x-y|} J_{\frac{x-y}{|x-y|}}(\rho|x-y|)\, \rho^2 d\rho.$$

where

$$J_v(k) = \int_{S^2} e^{\imath k(1-\omega\cdot v)} \left\{P_\pm(\rho\omega)z^*(x,\rho\omega)P_\pm(\rho\omega)z'(y,\rho\omega)\right\}\chi(\pm 2^{-j}\lambda(\rho\omega))\, d\omega.$$

We can suppose $v=(0;0;1)$ and so

$$J_v(k) = \int_0^{2\pi}\int_0^{\pi} e^{\imath k(1-\cos(\phi))}\Big\{P_\pm(\rho\omega(\theta,\phi))z^*(x,\rho\omega(\theta,\phi))\times \\ \times P_\pm(\rho\omega(\theta,\phi))z'(y,\rho\omega(\theta,\phi))\Big\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\sin(\phi)d\phi d\theta.$$

An integration by parts in ϕ gives

$$J_v(k) = \frac{1}{\imath k}\int_0^{2\pi}\Big[e^{\imath k(1-\cos(\phi))}\Big\{P_\pm(\rho\omega(\theta,\phi))z^*(x,\rho\omega(\theta,\phi))\times \\ \times P_\pm(\rho\omega(\theta,\phi))z'(y,\rho\omega(\theta,\phi))\Big\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\Big]_0^\pi\, d\theta \\ -\frac{1}{\imath k}\int_0^{2\pi}\int_0^{\pi} e^{\imath k(1-\cos(\phi))}\partial_\phi\Big\{P_\pm(\rho\omega(\theta,\cdot))z^*(x,\rho\omega(\theta,\cdot))\times \\ \times P_\pm(\rho\omega(\theta,\cdot))z'(y,\rho\omega(\theta,\cdot))\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\cdot)))\Big\}(\phi)\, d\phi d\theta.$$

The integrand of the first term can be rewritten in order to obtain a sum of two integral in ϕ over the interval $[0,\pi]$. To this end, we introduce a smooth cut-off function which splits $[0,\pi]$ in two parts one is a neighborhood of 0 and the other a neighborhood of π. Then most of the terms obtained after derivation can be treated by the method used for the $|t|^{-1}$ decay. Only the two terms where derivatives of z and z' appear need a particular treatment. Now we have to distinguish the case $z=z'=w$ from the two others where $z=u$ or $z'=u$. If $z=z'=w$, the terms which need a particular treatment are bounded by $C|t|^{-1}$ times the supremum in ϕ' of the $L^1_{\phi,\,\theta}([0,\pi]\times[0,2\pi])$ of

$$L^\pm_{j,n,m}(t,x,y,\phi,\phi') = \int_{\mathbb{R}^+} e^{\mp\imath t\sqrt{\rho^2+m^2}-\imath\rho|x-y|\cos(\phi')}\left\{P_\pm(\rho\omega)\partial_\phi^n z^*(x,\rho\omega)\right\}\times \\ \times\left\{P_\pm(\rho\omega(\theta,\phi))\partial_\phi^m z'(y,\rho\omega(\theta,\phi))\right\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\rho d\rho,$$

with $n,m\in\mathbb{N}$ such that $n+m=1$. It is a sum of terms of the form

$$\int_{\mathbb{R}^+} e^{\imath t\left\{\mp\sqrt{\rho^2+m^2}-\rho\frac{|x-y|}{t}\left(\cos(\phi')-\cos(\phi)\right)-\varepsilon_i\rho\frac{|x|}{t}+\varepsilon_{i'}\rho\frac{|y|}{t}\right\}}\times \\ \times\left\{P_\pm(\rho\omega)_{k,i}\left(e^{-\imath\psi(k,x)}\partial_\phi^n z^*(x,\rho\omega)\right)_{i,l}\right\}\times \\ \times\left\{P_\pm(\rho\omega(\theta,\phi))_{l,k'}\left(\partial_\phi^m z'(y,\rho\omega(\theta,\phi))e^{\imath\psi(k,y)}\right)_{k',i'}\right\}\chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi)))\,\rho d\rho.$$

where ϕ is the angle between $\frac{x-y}{|x-y|}$ and the $z-$axis and $\psi(x,k)\in\mathcal{M}_4(\mathbb{C})$ is given by

$$\begin{pmatrix}(|x||k|+x\cdot k)\,I_{\mathbb{C}^2} & 0_{\mathbb{C}^2}\\ 0_{\mathbb{C}^2} & (-|x||k|+x\cdot k)\,I_{\mathbb{C}^2}\end{pmatrix},$$

and $\varepsilon_i, {\varepsilon_i}' \in \{\pm 1\}$. We introduce

$$K(\rho) = \left\{ P_\pm(\rho\omega)_{k,i} \left(e^{-\imath\psi(k,x)} \partial_\phi^n z^*(x,\rho\omega) \right)_{i,l} \right\}$$

and

$$\left\{ P_\pm(\rho\omega(\theta,\phi))_{l,k'} \left(\partial_\phi^m z'(y,\rho\omega(\theta,\phi)) e^{\imath\psi(k,y)} \right)_{k',i'} \right\},$$

$$\phi(\rho) = \mp\sqrt{\rho^2+m^2} - \rho\frac{|x-y|}{t}\left(\cos(\phi') - \cos(\phi)\right) - \varepsilon_i \rho \frac{|x|}{t} + \varepsilon_{i'}\rho\frac{|y|}{t}$$

and

$$f(\rho) = \frac{\partial_\rho \phi(\rho)}{\inf\limits_{x \in \operatorname{supp}(\chi_j)} \left\{ |\partial_\rho^2 \phi(|x|,\lambda)| \right\}}.$$

With help of a smooth cut-off function, we split the integral in two parts. One has support $\{t \in \mathbb{R};\ |f(t)| \leq \alpha\}$ on which we use the estimate

$$\lambda\left(\{t \in \mathbb{R};\ |f(t)| \leq \alpha\}\right) \leq \alpha,$$

for λ the Lebesgue measure, since $|f'| > 1$. The other is its complementary, in which we make an integration by parts. We obtain the estimate

$$\begin{aligned} \left|J_j^+[r,r'](t,x,y)\right| &\leq C\alpha \max_{\rho \in A_j} \{\rho\,|K(\rho)|\} + \frac{1}{\alpha t \inf\limits_{\rho\in A_j}\left\{|\partial_\rho^2\phi(\rho)|\right\}} \times \\ &\times \max\left\{ \int_{A_j} \{\rho\,|(\partial_\rho K)(\rho)|\};\ \int_{A_j} |K(\rho)|\ d\rho;\ 2^{-j}\int_{A_j}\{\rho\,|K(\rho)|\} \right\}, \end{aligned}$$

where $A_j = g^{-1}\{\operatorname{supp}(\chi_j)\}$ with $g(\rho) = \sqrt{\rho^2+m^2}$. Hence with (3.21) of Proposition 3.6, Proposition 3.8 and decay of derivatives of $P_\pm$, we can choose $\alpha = 2^{2j}\sqrt{t}^{-1}$ and we obtain the bound of (3.24) in this case.
For the case $(z,z') = (u,w)$ (the case $(z,z') = (w,u)$ is similar), we study by the same way the integral

$$\begin{aligned} \int_{\mathbb{R}^+} & e^{\imath t\left\{\mp\sqrt{\rho^2+m^2} - \rho\frac{|x-y|}{t}\left(\cos(\phi')-\cos(\widetilde{\phi})\right) - \varepsilon_{i'}\rho\frac{|y|}{t}\right\}} \times \\ &\times \left\{ \left\{ P_\pm(\rho\omega)_{k,i}\left(\partial_\phi^n z^*(x,\rho\omega)\right)_{i,l}\right\} \times \right. \\ &\times \left\{ P_\pm(\rho\omega(\theta,\phi))_{l,k'}\left(\partial_\phi^m z'(y,\rho\omega(\theta,\phi))e^{\imath\psi(k,y)}\right)_{k',i'}\right\} \times \\ &\left. \times\, \chi(\pm 2^{-j}\lambda(\rho\omega(\theta,\phi))) \right\} \rho d\rho, \end{aligned}$$

where $\widetilde{\phi}$ is the angle between $\frac{y}{|y|}$ and the $z-$axis.

If $|x-y|/|t| \ll 1$, we can suppose $|x-y|/|t| < |\xi|/(2\sqrt{\xi^2+m^2})$ for any $\xi \in \operatorname{supp}(\chi_j)$ and instead of applying the trick of the proof of Lemma 3.1 (integration

by parts with respect to an angular variables) to the integral $I_j^{\pm}[z,z'](t,x,y)$, we make an integration by parts with help of

$$\frac{\partial_{|\xi|}}{\frac{|\xi|}{\sqrt{\xi^2+m^2}} \pm \frac{\xi}{|\xi|}\cdot\frac{x-y}{t}}.$$

The rest of the proof is the same.

We now turn to the proof of estimate (3.25), the kernel of the operator is given by a sum of terms of the form

$$I_j^{\pm}[z,z'](t,x,y) = \int_{\mathbb{R}^3} e^{\mp\imath t\sqrt{\xi^2+m^2}} e^{-\imath\xi\cdot(x-y)}\times \\ \times \{P_{\pm}(\xi)z^*(x,\xi)P_{\pm}(\xi)z'(y,\xi)\}\,\chi(\pm\lambda(\xi))\,d\xi.$$

We first notice that Proposition 3.6 implies that this integral is bounded. Then we split the integral in two parts. One is supported in a small neighborhood of the critical point of the phase, the other is its complementary. To treat this last integral we work exactly like the case "$\frac{|x-y|}{t} \ll 1$", just mentioned above. For the other one, we apply the Morse lemma to reduced the study to

$$\int_{\mathbb{R}^3} e^{\mp\imath t\xi^2}\{P_{\pm}(f(\xi))z^*(x,f(\xi))P_{\pm}(f(\xi))z'(y,f(\xi))\}\,\widetilde{\chi}(f(\xi))\,d\xi = \\ \int_{S^2}\int_{\mathbb{R}^+}\rho e^{\mp\imath t\rho^2}\Big\{P_{\pm}(f(\rho\omega))z^*(x,f(\rho\omega))\times \\ \times P_{\pm}(f(\rho\omega))z'(y,f(\rho\omega))\Big\}\widetilde{\chi}(f(\rho\omega))\,d\rho d\omega,$$

where $\widetilde{\chi}$ is the product of an indicator of a small neighborhood of the critical point with $\chi(\pm\lambda(\cdot))$. Then an integration by parts in ρ and the Van Der Corput lemma give (3.25) when $\theta = 1$. Since we have that the integral is bounded the general case easily follows.

We are now able to write the proof of Theorem 1.2, using Proposition 3.9.

Proof (Proof of Theorem 1.2). We notice that

$$\begin{aligned}\phi_k(D_m)\phi_j(H) &= D_m^{-1}\phi_k(D_m)H\phi_j(H) - D_m^{-1}\phi_k(D_m)V\phi_j(H)\\ &= 2^{-k}\widetilde{\phi}_k(D_m)2^j\widetilde{\phi}_j(H) - 2^{-k}\widetilde{\phi}_k(D_m)V\phi_j(H)\end{aligned}$$

We can also use H^{-1} since the support of ϕ_j is far from 0

$$\phi_k(D_m)\phi_j(H) = D_m\phi_k(D_m)H^{-1}\phi_j(H) - \phi_k(H)VH^{-1}\phi_j(H)$$

or higher power in D_m^{-1} or H^{-1} to obtain with (3.25)

$$\|\phi_i(D_m)e^{-\imath t(H)}\phi_j(H)\phi_k(D_m)\|_{L^1,\,L^\infty} \leq C2^{-r'|j-i|}\frac{C2^{(2+\theta)j}}{t^{1+\theta}}2^{-r|j-k|}$$

for any reals r, r' with C independent of i, j. Hence if r, $r' > 0$, we work like in the proof of Theorem 3.1 (*i.e.* like in Appendix 2 of [Bre77]) to conclude the proof.

It now remains to prove Proposition 3.6, 3.7 and 3.8.

3.3.3. Some estimates.

Estimates for w. We remind us of the definition of w in (3.19) and we introduce

$$\widetilde{R}_V^{\pm}(k) = e^{-ik\cdot Q} R_V^{+}(\pm\lambda(k)) e^{ik\cdot Q}. \tag{3.26}$$

We have

Lemma 3.3. *For any $\alpha \in \mathbb{N}^3$, let be $\sigma > 4 + |\alpha|$. Then there exists $C > 0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$*

$$\left|\left\{\nabla_k^\alpha \widetilde{R}_0^{\pm}(k)\langle Q\rangle^{-\sigma} q\right\}(x)\right| \le \frac{C}{\langle k\rangle^{|\alpha|}} \frac{\langle x\rangle^{|\alpha|}}{\left\langle |x| \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle} \|q\|_{W^{2+|\alpha|,\,\infty}}. \tag{3.27}$$

We also have that there exists $C > 0$ such that for any k, $x \in \mathbb{R}^3 \setminus \{0\}$

$$\begin{aligned}&\left|\left\{\nabla_k^\alpha \widetilde{R}_0^{\pm}(k)\langle Q\rangle^{-\sigma} q\right\}(x)\right| \\ &\qquad \le C \frac{\langle x\rangle^{\alpha-1}}{\langle k\rangle^{\alpha-1}} \frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\} \left|\frac{x}{|x|} - \frac{k}{|k|}\right|\right\rangle^2} \|q\|_{W^{2+|\alpha|,\,\infty}}. \end{aligned} \tag{3.28}$$

Proof. We write

$$\begin{aligned}&\left(\widetilde{R}_0^{\pm}(k)\langle Q\rangle^{-\sigma} q\right)(x) \\ &\quad = \int_{\mathbb{R}^3} \frac{e^{\imath\{\pm|k||y| + k\cdot y\}}}{4\pi|y|} \left\{\frac{\alpha\cdot(x-y)q(x-y)}{\langle x-y\rangle^{\sigma+2}} + \frac{\alpha\cdot\nabla q(x-y)}{\langle x-y\rangle^{\sigma}}\right\} dy \\ &\qquad + (\alpha\cdot k + m\beta \pm \lambda(k)) \int_{\mathbb{R}^3} \frac{e^{\imath\{\pm|k||y| + k\cdot y\}}}{4\pi|y|} \frac{q(x-y)}{\langle x-y\rangle^{\sigma}}\, dy.\end{aligned}$$

We restrict our study to $\widetilde{R}_0^{+}(k)$ since the two cases are similar. Hence we only need to estimate integrals of the form

$$R(k)(x) = \int_{\mathbb{R}^3} e^{\imath\{|k||y| + k\cdot y\}} \frac{u(x-y)}{|y|}\, dy$$

with $u \in W_\sigma^{1+|\alpha|,\,\infty}(\mathbb{R}^3, \mathbb{C})$.

In a first step, a straightforward calculation shows that

$$|\nabla_k^\alpha R(k)(x)| \le C\langle x\rangle^{|\alpha|-1} \|u\|_{L_\sigma^\infty} \tag{3.29}$$

if $\sigma > 3 + \max\{|\alpha| - 1; 0\}$. Then using the trick we used in the proof of Proposition 3.2, we obtain

$$\nabla_k^\alpha R(k)(x) = \frac{\imath^{|\alpha|}}{|k|^{|\alpha|}} \int_{\mathbb{R}^3} e^{\imath\{|k||y| + k\cdot y\}} \left\{(\nabla|Q|)^\alpha \frac{u(x-\cdot)}{|Q|}\right\}(y)\, dy,$$

and so with (3.29), we infer

$$|\nabla_k^\alpha R(k)(x)| \le \frac{C\langle x\rangle^{|\alpha|-1}}{\langle k\rangle^{|\alpha|}} \|u\|_{W_\sigma^{|\alpha|,\infty}}, \tag{3.30}$$

since $\sigma > 3 + \max\{|\alpha| - 1, 0\}$.

In a second step, we apply Estimate (3.5) of Proposition 3.2 to $R(k)(x)$, this gives

$$\begin{aligned} &|\nabla_k^\alpha R(k)(x)| \\ &\quad \le \frac{C}{|k|^{|\alpha|+1}} \max_{|\beta|\le 1+|\alpha|} \left\{ \int_{\mathbb{R}^3} |y|^{|\beta|-1} \frac{1}{\left|\frac{k}{|k|} - \frac{y}{|y|}\right|} \left| \nabla^\beta \left\{ \frac{u(x-\cdot)}{|Q|} \right\}(y) \right| \, dy \right\}. \end{aligned}$$

Hence we need to estimate on integral of the form

$$G(x,\omega) = \int_{\mathbb{R}^3} \frac{1}{|y|^n} \frac{1}{\langle x-y\rangle^s} \frac{1}{\left|\frac{y}{|y|} - \omega\right|} \, dy$$

with $\omega \in S^2$, $-|\alpha| + 1 \le n \le 2$ and $s > \sigma$. To obtain appropriate estimates, we use

$$|x-y| \ge \frac{1}{4} \max\{|y|, |x|\} \left| \frac{x}{|x|} - \frac{y}{|y|} \right| + \frac{1}{2} \left| |x| - |y| \right|$$

to write for $\theta, \theta' \ge 0$ such that $\theta + \theta' = 1$,

$$\begin{aligned} G(x,\omega) &\le C \int_{\mathbb{R}^3} \frac{1}{|y|^n} \frac{1}{\langle |x| - |y|\rangle^{\theta s}} \frac{1}{\left\langle |x| \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \right\rangle^{\theta' s}} \frac{1}{\left|\frac{y}{|y|} - \omega\right|} \, dy \\ &\le \frac{C}{|x| \left\langle |x| \left| \omega - \frac{x}{|x|} \right| \right\rangle} \int_{\mathbb{R}^+} \frac{1}{r^{n-2}} \frac{1}{\langle |x| - r\rangle^{\theta s}} \, dr \\ &\le \frac{C\langle x\rangle^{|\alpha|+1}}{|x| \left\langle |x| \left| \omega - \frac{x}{|x|} \right| \right\rangle}, \end{aligned}$$

if $\theta' s > 2$ and $\theta s > 1 + \max\{2 - n; 0\}$. Since $G(0,\omega)$ is bounded, we obtain

$$G(x,\omega) \le \frac{C\langle x\rangle^{|\alpha|}}{\left\langle |x| \left| \omega - \frac{x}{|x|} \right| \right\rangle},$$

Hence, we obtain with estimate (3.30)

$$|\nabla_k^\alpha R(k)(x)| \le \frac{C}{\langle k\rangle^{|\alpha|+1}} \frac{\langle x\rangle^{|\alpha|}}{\left\langle |x| \left| \frac{k}{|k|} - \frac{x}{|x|} \right| \right\rangle} \|u\|_{W_\sigma^{|\alpha|+1,\infty}}, \tag{3.31}$$

which gives estimate (3.27).

In a third step, if $k/|k| \ne x/|x|$, we split the integral for $\nabla_k^\alpha R(k)(x)$ in two parts with help of a smooth cut-off function defined in S^2 the support of which

is a half cone determined by the bisector plane of the couple $(k/|k|; x/|x|)$. So we obtain $\nabla_k^\alpha R(k)(x) = R_1(k)(x) + R_2(k)(x)$ with $R_1(k)(x)$ having a support containing $x/|x|$ and $R_2(k)(x)$ having a support containing $k/|k|$. We then apply the estimate (3.6) of Proposition 3.2 to $R_1(k)(x)$ to obtain

$$|\nabla_k^\alpha R_1(k)(x)| \leq \frac{C}{|k|^{|\alpha|+2}\left|\frac{k}{|k|}-\frac{x}{|x|}\right|}\times$$
$$\times \max_{|\beta|\leq 2+|\alpha|}\left\{\int_{\mathbb{R}^3}|y|^{|\beta|-2}\frac{1}{\left|\frac{k}{|k|}-\frac{y}{|y|}\right|}\left|\nabla^\beta\left\{\frac{u(x-\cdot)}{|Q|}\right\}(y)\right|\,dy\right\}.$$

This gives the estimate

$$|\nabla_k^\alpha R_1(k)(x)| \leq \frac{C}{|k|^{|\alpha|+2}}\frac{\langle x\rangle^{|\alpha|-1}}{\left|\frac{k}{|k|}-\frac{x}{|x|}\right|^2}\|u\|_{W_\sigma^{|\alpha|+2,\infty}},$$

since $\sigma > 2+|\alpha|$. Using (3.31), we infer

$$|\nabla_k^\alpha R_1(k)(x)| \leq \frac{C}{\langle k\rangle^{|\alpha|+1}}\frac{\langle x\rangle^{|\alpha|}}{\langle\sqrt{|k||x|}\left|\frac{k}{|k|}-\frac{x}{|x|}\right|\rangle^2}\|u\|_{W_\sigma^{|\alpha|+2,\infty}},$$

or, using (3.30),

$$|\nabla_k^\alpha R_1(k)(x)| \leq \frac{C}{\langle k\rangle^{|\alpha|}}\frac{\langle x\rangle^{|\alpha|-1}}{\langle|k|\left|\frac{k}{|k|}-\frac{x}{|x|}\right|\rangle^2}\|u\|_{W_\sigma^{|\alpha|+2,\infty}}.$$

For $R_2(x)(k)$, we use the inequality $|x-y| \geq \frac{|x|\left|\frac{x}{|x|}-\frac{y}{|y|}\right|}{2}$ to obtain

$$R_2(k)(x) \leq \frac{C\langle x\rangle^{|\alpha|-1}}{\langle k\rangle^{|\alpha|}\langle|x|\left|\frac{k}{|k|}-\frac{x}{|x|}\right|\rangle^s}\|u\|_{W_\sigma^{|\alpha|,\infty}},$$

since $\sigma > 3+s+\max\{|\alpha|-1,0\}$. So now we easily deduce estimate (3.28).

For the sequel, we need the following

Lemma 3.4. *Let be $s\in\mathbb{R}$ and ϕ a $\mathcal{C}^\infty$ function such that there is $\sigma>0$ with*

$$\forall\alpha\in\mathbb{N}^3,\ |\nabla^\alpha\phi(x)| \leq \frac{C_\alpha}{\langle x\rangle^\sigma}.$$

We have that $[\langle P\rangle^s, \phi(Q)]$ is bounded from H_q^t into $H_{q'}^{t'}$ with $q'+\sigma \geq q$ and $t'+1 \geq t+s$.

Proof. We want to prove that

$$\langle Q\rangle^q \langle P\rangle^t [\langle P\rangle^s, \phi(Q)]\langle P\rangle^{-t'}\langle Q\rangle^{-q'} \tag{3.32}$$

is bounded in $\mathcal{B}(L^2)$. Using the identity

$$[\langle P\rangle^s, \phi(Q)] = [\langle P\rangle^{s/2}, \phi(Q)]\langle P\rangle^{s/2} + \langle P\rangle^{s/2}[\langle P\rangle^{s/2}, \phi(Q)]$$

we reduce the proof to the case $|s| < 1$. And with the identity

$$[\langle P\rangle^s, \phi(Q)] = -\langle P\rangle^s[\langle P\rangle^{-s}, \phi(Q)]\langle P\rangle^s$$

we only need to study the case $-1 < s < 0$. The proof in this case is based on the following identity for $-1 < s < 0$

$$\langle P\rangle^s = (-\Delta+1)^{s/2} = \frac{-\sin(\pi\left\{\frac{s}{2}\right\})}{\pi}\int_0^{+\infty}\frac{w^{\left\{\frac{s}{2}\right\}}}{-\Delta+1+w}dw.$$

So we have

$$[\langle P\rangle^s, \phi(Q)] = \sum_{k=1}^m \frac{\Gamma(s/2+1)}{\Gamma(s/2+1-k)}(-\Delta+1)^{s/2-k} Ad^k_{-\Delta+1}(\phi(Q)) + R_m$$

with

$$R_m = \frac{(-1)^m\sin(\pi\left\{\frac{s}{2}\right\})}{\pi}\int_0^{+\infty}\frac{w^{\left\{\frac{s}{2}\right\}}}{(-\Delta+1+w)^{m+1}}Ad^{m+1}_{-\Delta+1}(\phi(Q))\frac{dw}{-\Delta+1+w}.$$

Then we use $\frac{-\Delta+1}{-\Delta+1+w} = 1 - \frac{w}{-\Delta+1+w}$, and we commute powers of $\langle P\rangle$ with operators of the form $\nabla^\alpha\phi(Q)$. Hence we can repeat the previous computation until we obtain only non positive powers of $\langle P\rangle$ in (3.32). So we only need to prove that operators of the form

$$[\langle Q\rangle^q, \phi(P)]\langle Q\rangle^{-q'}$$

with $q \le q'+1$ and ϕ satisfying the assumption of the lemma are bounded in $\mathcal{B}(L^2)$, we just repeat the previous calculation but we switch the role of P and Q. This ends the proof.

We now state a particular version of the Limiting Absorption Principle for H.

Proposition 3.10. *We assume that Assumptions 1.1 and 1.2 hold. Then for any $\sigma \ge 1$ there exists $C > 0$ such that for any $k \in \mathbb{R}^3$*

$$\|\widetilde{R}^{\pm}_V(k)\|_{\mathcal{B}(L^2_\sigma, L^2_{-\sigma})} \le C.$$

Proof. In fact, we just need to prove that for any $\sigma \ge 1$ there exists $C > 0$ such that for any $\lambda \in \mathbb{R}\setminus(-m, m)$

$$\|R^+_V(\lambda)\|_{\mathcal{B}(L^2_\sigma, L^2_{-\sigma})} \le C.$$

Using Theorem 1.1, we have that it is true if $\sigma > 5/2$. Then we use Born expansion

$$R^+_V(\lambda) = R^+_0(\lambda) - R^+_0(\lambda)VR^+_0(\lambda) + R^+_0(\lambda)VR^+_V(\lambda)VR^+_0(\lambda)$$

and [IM99, Theorem 2.1(i)] to end the proof.

We are now able to give the

Proof (Proof of Proposition 3.6). We only give a the general idea of the proof and we leave the details to the reader. We notice that with $\widetilde{R}_V^{\pm}$ defined by (3.26), we obtain

$$w = \widetilde{R}_V V u$$

with an abuse of notation since we avoid to distinguish the case where we have $\widetilde{R}_V^{+}$ or $\widetilde{R}_V^{-}$. We recall the identities

$$\widetilde{R}_V V = \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_V V = \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_0 + \widetilde{R}_0 V \widetilde{R}_V V \widetilde{R}_0 V. \tag{3.33}$$

Since, we have

$$\widetilde{R}_V = (1 + \widetilde{R}_0 V)^{-1} \widetilde{R}_0, \quad (1 + \widetilde{R}_0 V)^{-1} = 1 - \widetilde{R}_V V,$$

for $|\alpha| = 1$, we obtain

$$\nabla_k^\alpha \widetilde{R}_V = (1 - \widetilde{R}_V V) \nabla_k^\alpha \widetilde{R}_0 (1 - V \widetilde{R}_V). \tag{3.34}$$

Using (3.34), we obtain a formula where only derivatives of $\widetilde{R}_0$ appear (if there is derivatives). Then between a derivative of $\widetilde{R}_0$ and $\widetilde{R}_V$, we insert a $\widetilde{R_0}$ with the identity (3.33):

$$\widetilde{R}_V V \nabla_k^\alpha \widetilde{R}_0 V = \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V - \widetilde{R}_0 V \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V + \widetilde{R}_0 V \widetilde{R}_V V \widetilde{R}_0 V \nabla_k^\alpha \widetilde{R}_0 V.$$

This ensures that if $\rho > 5$, V or its derivatives decays enough to use Estimate (3.27) and Proposition 3.10. Since these estimates need derivatives and Sobolev's injections, we apply Lemma 3.4 to conclude the proof.

Estimates for v. We remind us of the definition of v in (3.22) and we introduce

$$S_V^{\varepsilon_1,\varepsilon_2}(k) = e^{-\varepsilon_1\varepsilon_2 \imath |k||Q|} R_V^{\varepsilon_1}(\varepsilon_2 \lambda(k)) e^{\imath k\cdot Q},$$

where $\varepsilon_i \in \{-1, 1\}$. With an abuse of notation, we will write $v = S_V V u$. We have

Lemma 3.5. *There exists $C > 0$, such that for any $k \in \mathbb{R}^3 \setminus \{0\}$ and $\beta \in \mathbb{N}^3$*

$$\left| \left(\nabla_k^\beta S_0^{\varepsilon_1,\varepsilon_2}(k) \langle Q\rangle^{-\sigma} q \right)(x) \right| \leq \frac{C}{\left\langle |x| \left| \frac{x}{|x|} - \frac{k}{|k|} \right| \right\rangle} \|q\|_{W^{2+|\beta|,\infty}}$$

for any $\sigma > 3 + |\beta|$.

Proof.

$$\begin{aligned}
&\left(S_0^{\varepsilon_1,\varepsilon_2}(k)\langle Q\rangle^{-\sigma} q\right)(x) = \\
&\quad \int_{\mathbb{R}^3} \frac{e^{\imath\varepsilon_1\varepsilon_2\{|k||x-y|-|k||x|+\varepsilon_1\varepsilon_2 k\cdot y\}}}{4\pi|x-y|} \left\{ \frac{\alpha\cdot y q(y)}{\langle y\rangle^{\sigma+2}} + \frac{\alpha\cdot\nabla q(y)}{\langle y\rangle^{\sigma}} \right\} dy \\
&\qquad + (\alpha\cdot k + m\beta \pm \lambda(k)) \int_{\mathbb{R}^3} \frac{e^{\imath\varepsilon_1\varepsilon_2\{|k||x-y|-|k||x|+\varepsilon_1\varepsilon_2 k\cdot y\}}}{4\pi|x-y|} \frac{q(y)}{\langle y\rangle^{\sigma}}\, dy.
\end{aligned}$$

For the sake of simplicity, we only write the proof when $\beta = 0$. The proof for derivatives works in the same way using $||x-y|-|x|| \leq |y|$ and $\sigma > 3 + |\beta|$. But the proof for the case $\beta = 0$, has been already done since

$$\left|\left(S_0^{\varepsilon_1,\varepsilon_2}(k)\langle Q\rangle^{-\sigma}q\right)(x)\right| = \left|\left(R_0^{\varepsilon_1,\varepsilon_2}(k)\langle Q\rangle^{-\sigma}q\right)(x)\right|.$$

Hence using Proposition 3.10, we able to write the

Proof (Proof of Proposition 3.7). We write with an abuse of notation

$$v = S_V V u,$$

and we use the Born formula

$$S_V V = S_0 V - S_0 V \widetilde{R}_V V,$$

together with Lemma 3.5, Propositions 3.4 and 3.10. The proof works like the one for w.

Estimates for $\widetilde{v}$. We remind us of the definition of $\widetilde{v}$ in (3.23) and we introduce

$$T_V^{\varepsilon_1,\varepsilon_2}(k) = e^{-\varepsilon_1\varepsilon_2 \imath |k||Q| + \imath k\cdot Q}\nabla_k \widetilde{R}_V^{\varepsilon_1}(\varepsilon_2\lambda(k))e^{\imath k\cdot Q},$$

where $\varepsilon_i \in \{-1, 1\}$. With another abuse of notation, here we will write $\widetilde{v} = T_V V u$. We have

Lemma 3.6. *There exists $C > 0$, such that for any $k \in \mathbb{R}^3 \setminus \{0\}$ and $\beta \in \mathbb{N}^3$*

$$\left|\left(\nabla_k^\beta T_0^{\varepsilon_1,\varepsilon_2}(k)\langle Q\rangle^{-\sigma}q\right)(x)\right| \leq C\frac{\langle \min\{|x|,\ |k|\}\rangle}{\langle k\rangle \left\langle \min\{|x|,\ |k|\}\left|\frac{x}{|x|}-\frac{k}{|k|}\right|\right\rangle^2}\|q\|_{W^{2+|\beta|,\infty}},$$

for any $\sigma > 4 + |\beta|$.

Proof. This is an obvious adaptation of the proof of Lemma 3.5, we just notice that one has

$$\left|\left(T_0^{\varepsilon_1,\varepsilon_2}(k)\langle Q\rangle^{-\sigma}q\right)(x)\right| = \left|\left(\nabla_k \widetilde{R}_V^{\varepsilon_1}(\varepsilon_2\lambda(k))\langle Q\rangle^{-\sigma}q\right)(x)\right|.$$

Hence, we have

Proof (Proof of Proposition 3.8). One more time, we write with an abuse of notation

$$v = T_V V u + S_V V \nabla_k u,$$

The second term of the right hand side could be studied exactly as we done in proof of Proposition 3.7 and for the first one we use the formula

$$T_V V = T_0 V - T_0 V \widetilde{R}_V V + S_0 V \nabla_k \widetilde{R}_V V,$$

together with Lemma 3.6, Propositions 3.4 and 3.10. The proof works like the one for w.

4. The linearized operator

In this section, we study the spectral properties of the linearized operator, associated with Equation (1.3), around a stationary state. This will be useful since we compare the dynamics associated with Equation (1.3) to the dynamic of the linear Dirac equation associated with H. This comparison is possible only because when the PLS is small, the linearized operator is a small perturbation of H.

4.1. The manifold of the particle like solutions. First we notice that Proposition 1.1, which gives the existence of stationary states, is a consequence of

Proposition 4.1. *Let H be a self adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and with a simple eigenvalue λ_0 associated with a normalized eigenvector ϕ_0. Assume that there is a neighborhood $\mathcal{O} \subset \mathbb{R}$ of λ_0 such that for all $\lambda \in \mathcal{O}$ the operator $(H - \lambda)^{-1}P_0$ is in $\mathcal{B}(L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4))$ for any $\sigma \in \mathbb{R}^+$, and in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ for any $l \in \mathbb{N}$, where P_0 is the projector into the orthogonal space of ϕ_0. Let $F \in \mathcal{C}^{k+1}(\mathbb{C}^4, \mathbb{C}^4)$ such that $F(z) = O(|z|^3)$.*
Then for any $\sigma \in \mathbb{R}^+$, there exists Ω a neighborhood of $0 \in \mathbb{C}$, a $\mathcal{C}^k$ map

$$h : \Omega \mapsto \{\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a $\mathcal{C}^k$ map $E : \Omega \mapsto \mathbb{R}$ such that $S(u) = u\phi_0 + h(u)$ satisfy for all $u \in \Omega$,

$$HS(u) + \nabla F(S(u)) = E(u)S(u),$$

with the following properties

$$\begin{cases} h(e^{i\theta}u) = e^{i\theta}h(u), \quad \forall \theta \in \mathbb{R}, \\ h(u) = O(|u|^2), \\ E(u) = E(|u|), \\ E(u) = \lambda_0 + O(|u|^2). \end{cases}$$

The proof of this proposition is an obvious adaptation of the one of [PW97, Proposition 2.2], and we don't repeat it here. One can also obtain it by means of the Crandall-Rabinowitz theorem but it doesn't give immediately the decomposition associated to the spectrum of $H = D_m + V$.
To show that $(H - \lambda)^{-1}P_0$ is in $\mathcal{B}(L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4))$ for any $\sigma > 0$, we just need to prove that $\alpha \mapsto e^{\alpha\langle Q\rangle}(H - \lambda)^{-1}P_0 e^{-\alpha\langle Q\rangle}$ is of class $\mathcal{C}^k$ near 0 in $\mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4))$ for any $k \in \mathbb{N}$, this can be proved with help of of [His00, Lemma 5.1]. To prove that $(H - \lambda)^{-1}P_0$ for any $l \in \mathbb{N}$ is in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ for any $l \in \mathbb{N}$, we first notice that $(D_m - \lambda)^{-1}$ is in $\mathcal{B}(H^l(\mathbb{R}^3, \mathbb{C}^4), H^{l+1}(\mathbb{R}^3, \mathbb{C}^4))$ then we use wave operator, see 3.12 and [GM01, Theorem 1.5], and the intertwining property, see 3.13, to conclude.
We shall need some properties of stationary solutions of (1.3). Following [His00], we have the

Lemma 4.1 (exponential decay). *For all $\beta \in \mathbb{N}^2$, $s \in \mathbb{R}^+$ and $p, q \in [1, \infty]$. There is $\gamma > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $u \in B_{\mathbb{C}}(0, \varepsilon)$ one has*

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S(u)\|_{B^s_{p,q}} \leq C\|S(u)\|_2,$$

where $\partial_u^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u \partial^{\beta_2}\Im u}$.

Proof. In fact we prove that for any k in $\mathbb{N}$ there is $\gamma > 0$ and $\varepsilon > 0$ and $C > 0$ such that for all $u \in B_{\mathbb{C}}(0, \varepsilon)$ one has

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S(u)\|_{H^k} \leq C\|S(u)\|_2.$$

Then interpolation and the following property of Besov spaces over $\mathbb{R}^3$ permit to conclude: $B^s_{2,2} = H^s$, $B^s_{p,r} \subset B^{s'}_{p,q}$ if $s' < s$, $B^u_{r,q} \subset B^s_{p,q}$ if $1 \leq r \leq p \leq \infty$ and $u - n/r = s - n/p$ and $\|uv\|_{B^s_{p,q}} \leq C\|u\|_{B^s_{q,t}}\|v\|_{B^s_{r,t}}$ if $\frac{1}{p} + \frac{s}{3} > \frac{1}{q} + \frac{1}{r}$.

We only prove the lemma for $\beta = 0$, the other cases are similar. We have

$$D_m S(u) + V S(u) + \nabla F(S(u)) = E(u)S(u).$$

Let us introduce the $\mathbb{R}-$linear operator W of multiplication by the matrix valued function $x \in \mathbb{R}^3 \mapsto -\imath D\nabla F(S(u)(x))\imath + V(x)$. We obtain, with the gauge invariance of F, the identity

$$WS(u) = \nabla F(S(u)) + VS(u).$$

The "potential" W tends to zero as x goes to ∞. In fact, as a function of x, W is in $L^1 \cap L^\infty$; we can write $W = W_c + W_\delta$ where W_c is compactly supported and $\|W_\delta\|_{L^1\cap L^\infty} \leq \delta$.

We have that $D_m + W_\delta - E(u)$ is invertible for δ sufficiently small and

$$e^{\gamma\langle Q\rangle}S(u) = e^{\gamma\langle Q\rangle}\left(D_m + W_\delta - E(u)\right)^{-1} e^{-\gamma\langle Q\rangle}\{e^{\gamma\langle Q\rangle}W_c S(u)\}.$$

For γ small, $\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)$ is invertible in L^2 and

$$e^{\gamma\langle Q\rangle}S(u) = \left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1} e^{\gamma\langle Q\rangle}W_c S(u).$$

This proves the lemma for $k = 0$ since $e^{\gamma\langle Q\rangle}W_c$ is bounded. Now we notice that

$$\begin{aligned}
&|P|\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1}\\
&= \frac{|P|}{D_m} - \frac{|P|}{D_m}\left(2\gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)\left(D_m + \gamma\frac{\alpha\cdot Q}{\langle Q\rangle} + W_\delta - E(u)\right)^{-1}.
\end{aligned}$$

Hence we obtain

$$\|e^{\gamma\langle Q\rangle}S(u)\|_{H^k} \leq C\|S(u)\|_{H^{k-1}}.$$

This identity proves the lemma by induction.

4.2. The spectrum of the linearized operator. Here we study the spectrum of the linearized operator associated with Equation (1.3) around a stationary state $S(u)$. Let us introduce

$$H(u) = H + d^2F(S(u)) - E(u)$$

where d^2F is the differential of ∇F. The operator $H(u)$ is $\mathbb{R}-$linear but not $\mathbb{C}-$linear. Replacing $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ with the inner product obtained by taking the real part of the inner product of $L^2(\mathbb{R}^3, \mathbb{C}^4)$, we obtain a symmetric operator. We then complexify this real Hilbert space and obtain $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ with its canonical hermitian product. This process transforms the operator $-\imath$ into

$$J = \begin{pmatrix} 0 & -Id_{\mathbb{C}^4} \\ Id_{\mathbb{C}^4} & 0 \end{pmatrix}.$$

For $\phi \in L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4) \subset L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$, we still write ϕ instead of

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}.$$

The extension of $H(u)$ over $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ is also written $H(u)$ and is now a real operator.

The linearized operator associated with Equation (1.3) around the stationary state $S(u)$ is given by $JH(u)$. We shall now study its spectrum. Differentiating (1.4), we have that

$$\mathcal{H}_0 = \operatorname{Span}\left\{\frac{\partial}{\partial \Re u} S(u),\ \frac{\partial}{\partial \Im u} S(u)\right\}$$

is invariant under the action of $JH(u)$. We notice (see [GNT04]) that

$$\mathcal{H}_0(u) = \operatorname{Span}\left\{JS(u), \partial_{|u|} S(u)\right\}.$$

Using gauge invariance and differentiating, we obtain

$$JH(u)JS(u) = 0 \quad \text{and} \quad JH(u)\partial_{|u|}S(u) = \partial_{|u|}E(u)JS(u).$$

Hence $\mathcal{H}_0(u)$ is contained in the geometric null space of $JH(u)$, in fact it is exactly the geometric null space as proved in the sequel of this subsection. First, we see that $JH(u)$ has two other simple eigenvalues, as stated in the following

Lemma 4.2. *Let be*

$$S_1^+(0) = \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \text{ and } S_1^-(0) = \begin{pmatrix} \overline{\phi_1} \\ i\overline{\phi_1} \end{pmatrix}.$$

Suppose that Assumptions 1.1–1.4 hold, then there are $\varepsilon > 0$ and four $\mathcal{C}^\infty$ maps $E_1^\pm : B_{\mathbb{C}}(0, \varepsilon) \mapsto \mathbb{C}$ and $k_1^\pm : B_{\mathbb{C}}(0, \varepsilon) \mapsto \left\{S_1^\pm(0)\right\}^\perp$ such that

$$JH(u)S_1^\pm(u) = E_1^\pm(u)S_1^\pm(u),$$

with $\|S_1^\pm(u)\| = 1$,

$$S_1^\pm(u) = S_1^\pm(0) + k_1^\pm(u),$$

with $E_1^\pm(u) = \pm i(\lambda_1 - \lambda_0) + O(|u|^2)$ and $k_1^\pm(0) = 0$.

Proof. This can be proved in the same fashion as [PW97, Proposition 2.2] using Assumption 1.3.

We also obtain

Lemma 4.3 (exponential decay in Besov spaces). *Suppose that Assumptions 1.1–1.4 hold, then for any $\beta \in \mathbb{N}^2$, $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ there is $\gamma > 0$, $\varepsilon > 0$ and a positive constant C such that for all $u \in B_{\mathbb{C}}(0, \varepsilon)$,*

$$\|e^{\gamma\langle Q\rangle}\partial_u^\beta S_1^\pm(u)\|_{B^s_{p,q}} \le C\|S_1^\pm(u)\|_2,$$

where $\partial_u^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u\partial^{\beta_2}\Im u}$.

Proof. The proof is exactly the same as the one of Lemma 4.1.

Let $\mathcal{H}_{\pm 1}(u)$ be the space spanned by $S_1^\pm(u)$. Let us now prove that the orthogonal space with respect to the hermitian product associated to J

$$\mathcal{H}_c(u) = \{\mathcal{H}_0(u) \oplus \mathcal{H}_{+1}(u) \oplus \mathcal{H}_{-1}(u)\}^\perp$$

contains no eigenvector. We notice that $\mathcal{H}_c(u)$ is invariant under the action of $JH(u)$. We have

Lemma 4.4 (Continuous subspace property). *If Assumptions 1.1–1.4 hold, let $\mathbf{P}_c(u)$ be the orthogonal projector onto $\mathcal{H}_c(u)$. Then there exists $\varepsilon > 0$ such that for u', $u \in B_{\mathbb{C}}(0, \varepsilon)$*

$$\mathbf{P}_c((u))|_{\mathcal{H}_c(u')} : \mathcal{H}_c(u') \mapsto \mathcal{H}_c(u)$$

is an isomorphism from $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^8) \cap \mathcal{H}_c(u')$ into $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^8) \cap \mathcal{H}_c(u)$, for any $s \in \mathbb{R}^+$ and any $p, q \in [1, \infty]$. The inverse $R(u', u)$ is continuous with respect to u and u'.

Proof. This proof is a straightforward adaptation of the one of [GNT04, Lemma 2.2].

So we have

Lemma 4.5. *Under the assumptions of Proposition 1.1, there exists $\varepsilon > 0$ such that for any $u \in B_{\mathbb{C}}(0, \varepsilon)$*

$$\begin{aligned} \|\langle Q\rangle^{-\sigma} e^{sJH(u)}\mathbf{P}_c(u)\psi\| &\le C\langle s\rangle^{-\min\{\sigma,\, 3/2\}}\|\langle Q\rangle^\sigma\psi\|, \ \forall\psi \in L^2_\sigma \\ \int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{sJH(u)}\mathbf{P}_c(u)\psi\|^2\, ds &\le C\|\psi\|, \ \forall\psi \in L^2. \end{aligned}$$

As a consequence, $\mathcal{H}_c(u)$ does not contain any eigenvector.

Proof. For the sake of clarity, we introduce

$$\zeta(u) = \left(J\frac{\partial}{\partial\Re u}S(u),\ J\frac{\partial}{\partial\Im u}S(u),\ JS_1^+(u),\ JS_1^-(u)\right).$$

Writing Duhamel's formula for $H(u)$ with respect to $H-E(u)$, we obtain

$$e^{tJH(u)}\mathbf{P}_c(u) = e^{tJ(H-E(u))}\mathbf{P}_c(u) \\ + \int_0^t e^{(t-s)J(D_m+V-E(u))}Jd^2F(S(u))e^{sJH(u)}\mathbf{P}_c(u)\,ds.$$

We have

$$\mathbf{P}_c(0)|^{-1}_{\mathcal{H}_c(u')} = R(u',0) = Id_{L^2} + \sum_i |\alpha_i(u',0)\rangle\,\langle\zeta_i(0)|$$

where the coordinates of $\alpha_i(u',u)$ are linear combination of the coordinates of $\zeta(u)$, so it can be extended to $L^2_{-\sigma}$ and we have

$$\begin{aligned}
&\|\langle Q\rangle^{-\sigma}e^{-tJH(u)}\mathbf{P}_c(u)\psi\| \\
&\leq \|\langle Q\rangle^{-\sigma}\ \mathbf{P}_c(0)|^{-1}_{\mathcal{H}_c(u')}\,\langle Q\rangle^{\sigma}\|\Big\{\|\langle Q\rangle^{-\sigma}\mathbf{P}_c(0)e^{-tJ(D_m+V-E(u))}\mathbf{P}_c(u)\psi\| \\
&\quad + \int_0^t \Big\|\langle Q\rangle^{-\sigma}\mathbf{P}_c(0)e^{-J(t-s)(D_m+V-E(u))}JD\nabla F(S(u))e^{-sJH(u)}\mathbf{P}_c(u)\psi\Big\|\,ds\Big\} \\
&\leq C\langle t\rangle^{-\min\{\sigma,\,3/2\}}\|\langle Q\rangle^{\sigma}\psi\| \\
&\quad +C\int_0^t \langle t-s\rangle^{-\min\{\sigma,\,3/2\}}\|\langle Q\rangle^{2\sigma}D\nabla F(S(u))\|\|\langle Q\rangle^{-\sigma}e^{-isH(u)}\mathbf{P}_c(u)\psi\|\,ds
\end{aligned}$$

We then introduce

$$M(t) = \sup_{s\in[0,t]}\{\langle s\rangle^{-\min\{\sigma,\,3/2\}}\|\langle Q\rangle^{-\sigma}e^{-sJH(u)}\mathbf{P}_c(u)\psi\|\}$$

and we obtain for $|z|\leq\varepsilon$

$$M(t)\leq C(\|\langle Q\rangle^{\sigma}\psi\| + \varepsilon M(t))$$

which gives for ε sufficiently small

$$M(t)\leq C\|\langle Q\rangle^{\sigma}\psi\|,$$

or

$$\|\langle Q\rangle^{-\sigma}e^{-sJH(u)}\mathbf{P}_c(u)\psi\| \leq C\langle s\rangle^{-\min\{\sigma,\,3/2\}}\|\langle Q\rangle^{\sigma}\psi\|.$$

With the same method, see Lemma A.2, we obtain the second estimate.
Then we obtain with the second estimate that there is no stationary state in the range of $\mathbf{P}_c(u)$ that is to say $\mathcal{H}_c(u)$.

This gives

Lemma 4.6. *We have, for sufficiently small $u\in\mathbb{C}$, $E_1^{\pm}(u)\in\imath\mathbb{R}$ with $E_1^{\pm}(u) = -E_1^{\mp}(u)$ and $S_1^{-}(u) = \overline{S_1^{+}(u)}$ for the conjugation of $\mathbb{C}^8$.*

Proof. The last statement straightforwardly follows from

$$JH(u)S_1^{\pm}(u) = E_1^{\pm}(u)S_1^{\pm}(u),$$

since there is no more eigenvalues than the 0 and $E_1^{\pm}(u)$, we obtain $\overline{E_1^{\pm}(u)} = E_1^{\mp}(u)$.

Then we specify the essential spectrum of $JH(u)$. A classical study gives that the continuous spectrum of $JH(0)$ is given by

$$\{\imath\lambda;\ \lambda \in \mathbb{R},\, |\lambda| \geq \min\{|m-\lambda_0|, |m+\lambda_0|\}\}\,.$$

Using Weyl's criterion (see [RS78, Theorem XIII.14, Corollary 1], the adaptation is quite easy in our case), we obtain that the essential spectrum is

$$\{\imath\lambda;\ \lambda \in \mathbb{R},\, |\lambda| \geq \min\{|m-E|, |m+E|\}\}\,.$$

Hence $E_1^{\pm}(u)$ are necessarily purely imaginary. Indeed if $H(u) - E_1^{\pm}(u)J$ is not invertible then $H(u) + \overline{E_1^{\pm}(u)}J$ is not invertible too. Since $-\overline{E_1^{\pm}(u)}$ is not in the essential spectrum, it is necessarily an eigenvalue in the neighborhood of $\pm\imath(\lambda_1 - \lambda_0)$. Hence this gives $-\overline{E_1^{\pm}(u)} = E_1^{\pm}(u)$.

4.3. Decomposition of the system. We want to decompose a solution ϕ of the equation (1.3) with respect to the spectrum of $JH(u)$. And in fact, we only study the resulting equations for these different parts of the decomposition. First we isolate a part which corresponds to a PLS. For any solution of (1.3) over an interval of time I containing 0, we write for $t \in I$

$$\phi(t) = e^{-\imath\int_0^t E(u(s))\,ds}\,(S(u(t)) + \eta(t))\,.$$

In order to give an equation for η, we introduce the following space

$$\mathcal{H}_0^{\perp}(u) = \left\{\eta \in L^2(\mathbb{R}^3, \mathbb{C}^8), \left\langle J\eta, \frac{\partial}{\partial \Re u}S(u)\right\rangle = 0,\ \left\langle J\eta, \frac{\partial}{\partial \Im u}S(u)\right\rangle = 0\right\}.$$

In fact it is the space

$$\mathcal{H}_{+1}(u) \oplus \mathcal{H}_{-1}(u) \oplus \mathcal{H}_c(u)$$

which is invariant under the action of $JH(u)$ and we state the

Lemma 4.7 (decomposition lemma). *Let be $s \geq 0$ and $p \geq 1$ there exist $\delta > 0$ and a $\mathcal{C}^\infty$ map $U : B_{W^{s,p}}(0,\delta) \mapsto B_{\mathbb{C}}(0,\varepsilon)$ which satisfies for $\psi \in B_{W^{s,p}}(0,\delta)$*

$$\psi = S(u) + \eta,\ \textit{with}\ \eta \in \mathcal{H}_0^{\perp}(u) \Longleftrightarrow u = U(\psi).$$

Proof. It is [GNT04, Lemma 2.3].

This lemma ensures that we can impose the orthogonality condition

$$\eta(t) \in \mathcal{H}_0^{\perp}(u(t)). \tag{4.1}$$

So instead of solving the Equation (1.3) in ϕ, we want to solve the equation

$$\begin{aligned} \imath\partial_t\eta &= \{H - E(u)\}\,\eta + \{\nabla F(S(u)+\eta) - \nabla F(S(u))\} - \imath dS(u)\dot{u} \\ &= \left\{H + d^2F(S(u)) - E(u)\right\}\eta + N(u,\eta) - \imath dS(u)\dot{u} \end{aligned} \tag{4.2}$$

for $\eta \in \mathcal{H}_0^{\perp}(u(t))$. Here d^2F is the differential of ∇F and dS the differential of S in $\mathbb{R}^2$. To close the system, we need an equation for u. Let us now derive an equation for the path u, by means of (4.1):

$$\langle \eta(t), JdS(u(t))\rangle = 0.$$

After a time derivation, we obtain

$$\begin{aligned} 0 = \langle JH(u(t))\eta(t) + JN(u(t)), \eta(t)) \\ + dS(u(t))\dot{u}(t), JdS(u(t))\rangle - \langle \eta, Jd^2S(u(t))\dot{u}(t)\rangle. \end{aligned}$$

Since $S(u) \in J\mathcal{H}_0(u)$, we have

$$\langle H(u)\eta, dS(u)\rangle = \langle \eta, H(u)dS(u)\rangle = \langle \eta, dE(u)S(u)\rangle = 0,$$

we obtain

$$[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle]\dot{u}(t) = -\langle N(u(t),\eta(t)), dS(u(t))\rangle.$$

So we notice that

$$[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O(|u(t)| + \|\eta(t)\|_2),$$

which proves that $[\langle JdS(u(t)), dS(u(t))\rangle - \langle J\eta(t), d^2S(u(t))\rangle]$ is invertible for small $|u(t)|$ and $\|\eta(t)\|_2$, we therefore introduce its inverse

$$A(u,\eta) = [\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}$$

and write

$$\partial_t u(t) = -A(u(t),\eta(t))\langle N(u(t),\eta(t)), dS(u(t))\rangle.$$

Plugging in Equation (4.2), and similarly to the linear case we decompose η with respect to the spectral decomposition of $H(u) = H + D\nabla F(S(u)) - E(u)$

$$\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$$

with $z \in \mathcal{H}_c(u) \cap L^2(\mathbb{R}^3, \mathbb{R}^8)$ and $\alpha^- = \overline{\alpha^+}$. We obtain the system

$$\left\{\begin{array}{l} \dot{u} \;\; = -A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ \dot{\alpha}^{\pm} = E^{\pm}(u)\alpha^{\pm} + \langle JN(u,\eta), JS_1^{\pm}(u)\rangle \\ \qquad\quad +\langle dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle\, JS_1^{\pm}(u)\rangle \\ \qquad\qquad -\langle (dS_1^{\pm}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\pm} \\ \qquad\qquad -\langle (dS_1^{\mp}(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^{\pm}(u)\rangle\alpha^{\mp} \\ \partial_t z = JH(u)z + \mathbf{P}_c(u)JN(u,\eta) \\ \qquad\quad +\mathbf{P}_c(u)dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ \qquad\qquad -(D\mathbf{P}_c(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle\eta \end{array}\right. ,$$

which we will now study. We notice that this equation is defined only for z small with real values, $\alpha^- = \overline{\alpha^+}$ small and u small.

5. The stabilization towards the PLS manifold

We now build a solution which stabilizes towards the manifold of the stationary states. To this end, we will use Theorem 1.1 and Theorem 1.2 to prove that z tends to zero in L^∞ and L^2_{loc}. It is possible here since we build solutions for which we ensure that α^+ and α^- also tend to zero. We do not think that this convergence holds for all initial states but we do not know any counterexample. We also notice that we look for a real solution $\phi = S(u) + \eta$, hence η should be real and therefore $\alpha^- = \overline{\alpha^+}$.
We impose the following condition

$$|\alpha| \le \frac{C}{\langle t\rangle^2}.$$

Under the assumptions of Theorem 1.3, let us define for any ε, $\delta > 0$

$$\mathcal{U}(\varepsilon,\delta) = \Bigg\{ u \in \mathcal{C}^\infty(\mathbb{R}, B_{\mathbb{C}}(0,\varepsilon)),\ \lim_{t\to+\infty} u(t) = u_\infty \text{ exists}, \qquad |u(t) - u_\infty| \le \frac{\delta^2}{\langle t\rangle^2}, \forall t \ge 0 \Bigg\}$$

and for any $u \in \mathcal{U}(\varepsilon)$, let s, s', β be such that $s' \ge s + 3 \ge \beta + 6$ and $\sigma > 5/2$, we define

$$\mathcal{Z}(u,\delta) = \Bigg\{ z \in \mathcal{C}^\infty(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(u(t)), \max\Bigg[\sup_{v\in[0,+\infty]}\{\|z(v)\|_{H^{s'}}\}, \sup_{v\in[0,+\infty]}\{\langle v\rangle^{3/2}\|z(v)\|_{B^{\beta}_{\infty,2}}\}, \sup_{v\in[0,+\infty]}\{\langle v\rangle^{3/2}\{\|z(v)\|_{H^s_{-\sigma}}\}\Bigg] < \delta \Bigg\}.$$

Then we define the space

$$\Omega(\delta) = \left\{ \alpha = \left(\alpha^+, \alpha^-\right) \in \mathcal{C}^\infty(\mathbb{R}),\ \alpha^- = \overline{\alpha^+},\ \sup_{t\in\mathbb{R}^+} \langle t\rangle^{3/2}|\alpha(t)| < \delta^2 \right\}.$$

5.1. Step 1: Construction of α. For any $u \in \mathcal{U}(\varepsilon,\delta)$ and $z \in \mathcal{Z}(u,\delta)$, let us define a map $\mathcal{G}_{u,z}$ on $\Omega(\delta)$ by

$$\begin{aligned}\mathcal{G}_{u,z}(\alpha)^\pm(t) = -\int_t^\infty e^{\int_s^t E_1^\pm(u(w))\,dw} \Bigg\{ &\langle JN(u(v),\eta(v)), S_1^\pm(u(v))\rangle \\ &+ \langle dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\, S_1^\pm(u(v))\rangle \\ &- \langle (dS_1^\pm(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\ S_1^\pm(u(v))\rangle\alpha^\pm(v) \\ &- \langle (dS_1^\mp(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\ S_1^\pm(u(v))\rangle\alpha^\mp(v) \Bigg\}\, dv.\end{aligned}$$

We want to show that $\mathcal{G}_{u,z}$ stabilizes $\Omega(\delta)$ and is a contraction for the L^∞ norm. We have the

Lemma 5.1. *Let be $\sigma \in \mathbb{R}$, $s > 1$ and $p, p_1, p_2, q \in [1, \infty]$ such that*

$$\frac{1}{p} + \frac{s}{3} > \frac{1}{p_1} + \frac{1}{p_2}.$$

Then there exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $u \in B_{\mathbb{C}}(0, \varepsilon_0)$ and $\eta \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8) \cap L^\infty(\mathbb{R}^3, \mathbb{R}^8)$, such that

$$\|\langle Q\rangle^\sigma N(u,\eta)\|_{B^s_{p,q}} \leq C\left(|u| + \|\eta\|_{B^s_{p_2,q}}\right)\|\eta\|_{L^\infty}\|\langle Q\rangle^\sigma \eta\|_{B^s_{p_1,q}}. \tag{5.1}$$

Proof. We recall the definition

$$N(u,\eta) = \nabla F(S(u)+\eta) - \nabla F(S(u)) - d^2F(S(u))\eta.$$

We have

$$N(u,\eta) = \int_0^1 \int_0^1 d^3F(S(u) + \theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta.$$

Since for $s \in \mathbb{R}^*_+$, p, p_1, p_2, $\in [1,\infty]$ such that $\frac{1}{p} + \frac{s}{3} > \frac{1}{p_1} + \frac{1}{p_2}$, we have $\|uv\|_{B^s_{p,q}} \leq C\|u\|_{B^s_{p_1,q'}}\|v\|_{B^s_{p_2,q'}}$. Then since $s > 1$, we use (see [EV97, Proposition 2.1])

$$\|d^3F(\psi)\|_{B^s_{p',q}} \leq C\left(s, F, \|\psi\|_\infty\right)\|\psi\|_{B^s_{p',q}}.$$

Then using Lemma 4.1, we conclude the proof.

Hence we have the

Lemma 5.2. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$, for any $u \in \mathcal{U}(\varepsilon, \delta)$ and $z \in \mathcal{Z}(u, \delta)$, $\mathcal{G}_{u,z}(\alpha)$ maps $\Omega(\delta)$ into itself.*

Proof. We have by means of Estimate (5.1) with e.g. $\sigma < -3$, $s = 0$, $p = q = 2$ and $p_1 = p_2 = 4$, if $u_0 \in \mathbb{C}$ and $z_0 \in \mathcal{H}_c(u_0) \cap H^{s'}_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ are small enough

$$\begin{aligned}
&\left|\mathcal{G}_{u,z}(\alpha)^\pm(t)\right| \\
&\leq C\int_t^\infty \Big\{ \left|\langle JN(u(s),\eta(s)), JS_1^\pm(u(s))\rangle\right| \\
&+ \left|\langle dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\, JS_1^\pm(u(v))\rangle\right| \\
&+ \left|\langle (dS_1^\pm(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\ JS_1^\pm(u(v))\rangle\alpha^\pm(v)\right| \\
&+ \left|\langle (dS_1^\mp(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle,\ JS_1^\pm(u(v))\rangle\alpha^\mp(v)\right|\Big\}\, dv \\
&\leq C\delta^2\langle t\rangle^{-2}.
\end{aligned}$$

Hence for small δ and small ε, we have $\mathcal{G}_{u,z}(\Omega(\delta)) \subset \Omega(\delta)$.

To prove that $\mathcal{G}_{u,z}$ is a contraction for the L^∞ norm, we use the

Lemma 5.3. *Let be $\sigma \in \mathbb{R}$, $s > 0$ and $p, q \in [1, \infty]$ such that $sp > 3$. Then for any $\varepsilon > 0$ and $M > 0$ there exists $C > 0$ such that for all u, $u' \in B_{\mathbb{C}}(0, \varepsilon)$ and η, $\eta' \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8) \cap L^\infty(\mathbb{R}^3, B_{\mathbb{R}^8}(0, M))$, such that*

$$
\begin{aligned}
\|\langle Q\rangle^{\sigma}\left\{N(u,\eta)-N(u',\eta')\right\}\|_{B^s_{p,q}} \leq C\Bigg\{ &\left(\|\langle Q\rangle^{\sigma_1}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_1}\eta'\|_{B^s_{p,q}}\right)^2 \times \\
&\times\left(|u-u'|+\|\langle Q\rangle^{\sigma_2}(\eta-\eta')\|_{B^s_{p,q}}\right) \\
&+\left(|u|+|u'|+\left\|\langle Q\rangle^{\sigma_1'}\eta\right\|_{B^s_{p,q}}+\left\|\langle Q\rangle^{\sigma_1'}\eta'\right\|_{B^s_{p,q}}\right)\times \\
&\times\left(\|\langle Q\rangle^{\sigma_2'}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_2'}\eta'\|_{B^s_{p,q}}\right)\|\langle Q\rangle^{\sigma_3'}(\eta-\eta')\|_{B^s_{p,q}}\Bigg\},
\end{aligned}
$$

with $2\sigma_1 + \sigma_2 = \sigma_1' + \sigma_2' + \sigma_3' = \sigma$.

Proof. Since, we have

$$N(u,\eta) = \int_0^1\int_0^1 d^3F(S(u)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta.$$

we can also restrict the study to $d^3F(\phi) - d^3F(\phi')$. If $d^5F \neq 0$, we have

$$
\begin{aligned}
&\|\langle Q\rangle^{\sigma}\left(d^3F(\phi)-d^3F(\phi')\right)\|_{B^s_{p,q}} \\
&\qquad\leq \int_0^1 \|d^4F(\phi+t(\phi-\phi'))\|_{B^s_{p,q}}\|\langle Q\rangle^{\sigma}(\phi-\phi')\|_{B^s_{p,q}}\,dt
\end{aligned}
$$

then since $s > 0$, we use

$$\|d^4F(\psi)\|_{B^s_{p,q}} \leq C(s, F, \|\psi\|_{B^s_{p,q}}).$$

Then using Lemma 4.1, we conclude the proof when $d^5F \neq 0$. Otherwise the proof is easily adaptable since d^4F is a constant matrix of $\mathcal{M}_4(\mathbb{C})$.

We also need the

Lemma 5.4. *Let be $\sigma \in \mathbb{R}$, $s > 0$ and $p, q \in [1, \infty]$. For any $\varepsilon > 0$ and $M > 0$, there exists $C > 0$ such that for all u, $u' \in B_{\mathbb{C}}(0, \varepsilon)$ and η, $\eta' \in B^s_{p,q}(\mathbb{R}^3, \mathbb{R}^8) \cap L^\infty(\mathbb{R}^3, B_{\mathbb{R}^8}(0, M))$, one has*

$$|A(u,\eta)-A(u',\eta')| \leq C\left\{|u-u'|+\|\langle Q\rangle^{\sigma}\{\eta-\eta'\}\|_{B^s_{p,q}}\right\}$$

Proof. We recall that

$$A(u,\eta) = [\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}.$$

We have

$$
\begin{aligned}
&A(u,\eta)-A(u',\eta') = -[\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle]^{-1}\times \\
&\times\left\{\langle JdS(u), dS(u)\rangle - \langle J\eta, d^2S(u)\rangle - \langle JdS(u'), dS(u')\rangle + \langle J\eta', d^2S(u')\rangle\right\}\times \\
&\qquad\qquad\times[\langle JdS(u'), dS(u')\rangle - \langle J\eta', d^2S(u')\rangle]^{-1}.
\end{aligned}
$$

The lemma then follows from Lemma 4.1.

Hence we have the

Lemma 5.5. *There exists $\delta_0 > 0$ and ε_0 such that there exists $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,\varepsilon_0)$, for any u, $u' \in \mathcal{U}(\varepsilon)$ and $z \in \mathcal{Z}(u,\delta)$ and $z' \in \mathcal{Z}(u',\delta)$, for any $\alpha,\alpha' \in \Omega(\delta)$, one has*

$$\begin{aligned}\|\mathcal{G}_{u',z'}(\alpha') - \mathcal{G}_{u,z}(\alpha)\|_{L^\infty(\mathbb{R}^+)}& \\ \leq \kappa \Big(|u'-u|_{L^\infty(\mathbb{R}^+)} &+ |\alpha'-\alpha|_{L^\infty(\mathbb{R}^+)} + \|z'-z\|_{L^\infty(\mathbb{R}^+,B^\beta_{\infty,2})}\Big).\end{aligned}$$

Proof. It is a straightforward computation based on Lemma 5.3 with e.g. $\sigma < -6$, $\sigma_2, \sigma_3' < -3$ and $s = 0$, $p = q = 2$, on Lemma 5.4, on Lemma 5.1 with e.g. $\sigma < -3$, $p = q = 2$ and $p_1 = p_2 = 4$ and on Lemma 4.3.

We now state the

Lemma 5.6. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for $\delta \in (0,\delta_0)$ and $\varepsilon \in (0,\varepsilon_0)$ and any $u \in \mathcal{U}(\varepsilon)$ and $z \in \mathcal{Z}(u,\delta)$, the equation*

$$\begin{aligned}\dot{\alpha}^\pm = E^\pm(u)\alpha^\pm &+ \langle JN(u,\eta), JS_1^\pm(u)\rangle \\ &+ \langle dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle\, JS_1^\pm(u)\rangle \\ &- \langle (dS_1^\pm(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^\pm(u)\rangle\alpha^\pm \\ &- \langle (dS_1^\mp(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle,\ JS_1^\pm(u)\rangle\alpha^\mp,,\end{aligned}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$, has a unique solution in $\Omega(\delta)$.

Proof. The proof is now classical since we proved that the integral equation

$$\alpha(t) = \mathcal{G}_{u,z}(\alpha)(t)$$

can be solved by means of the fixed point theorem.

5.2. Step 2: Construction of z. Let be $u \in \mathcal{U}(\varepsilon,\delta)$ and $z_0 \in \mathcal{H}_c(u(0)) \cap H^{s'}_\sigma$. Let us write $u_\infty = \lim_{t\to+\infty} u(t)$, we define $\mathcal{T}_{u,z_0}(z)$ by

$$\begin{aligned}\mathcal{T}_{u,z_0}(z)(t) &= e^{JtH(u_\infty)}z_0 \\ &- \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left\{E(S(u(v))) - E(S(u_\infty))\right\}z(v)\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left\{d^2F(S(u(v))) - d^2F(S(u_\infty))\right\}z(v)\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv \\ &+ \int_0^t e^{J(t-v)H(u_\infty)}\mathbf{P}_c(u(v))dS(u(s))A(u(v),\eta)\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv \\ &- \int_0^t e^{J(t-v)H(u_\infty)}(d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\eta(v)\,dv.\end{aligned}$$

We have

Lemma 5.7. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ and $C > 0$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$ and for any $u \in \mathcal{U}(\varepsilon,\delta)$, the application $\mathcal{T}_{u,z_0}$ maps $\mathcal{Z}(u,\delta)$ into itself if $\|z_0\|_{H^{s'}_\sigma} \le C\delta$*

Proof. With Lemma A.1 and Lemma 4.4, we obtain

$$\begin{aligned}
&\|\mathcal{T}_{u,z_0}(t)\|_{H^{s'}}\\
&\le C\|\mathbf{P}_c(u_\infty)\mathcal{T}_{u,z_0}(t)\|_{H^{s'}}\\
&\le C\|z_0\|_{H^{s'}} + C\int_0^t \|\{E(S(u(v))) - E(S(u_\infty))\}\, z(v)\|_{H^{s'}}\, dv\\
&\quad + C\int_0^t \left\|\left\{d^2F(S(u(v))) - d^2F(S(u_\infty))\right\} z(v)\right\|_{H^{s'}}\, dv\\
&\quad + C\int_0^t \|N(u(v),\eta(v))\|_{H^{s'}}\, dv\\
&\quad + C\int_0^t \|dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\|_{H^{s'}}\, dv\\
&\quad + C\int_0^t \|(d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\eta(v)\|_{H^{s'}}\, dv.
\end{aligned}$$

Now, with Lemma 5.1, we obtain

$$\begin{aligned}
\|\mathcal{T}_{u,z_0}(t)\|_{H^{s'}} \le{}& C\|z_0\|_{H^{s'}} + C\varepsilon\int_0^t |u(v) - u_\infty|\, \|z\|_{H^{s'}}\, dv\\
&+ C\int_0^t \left(|u(v)| + \|\eta(v)\|_{H^{s'}}\right) \|\eta(v)\|_{L^\infty}\, \|\eta(v)\|_{H^{s'}}\, dv\\
&+ C\int_0^t \left(|u(v)| + \|\eta(v)\|_{H^{s'}}\right) \|\eta(v)\|_{L^\infty}\, \|\eta(v)\|_{H^{s'}}\, \|\eta(v)\|_{H^{s'}}\, dv,
\end{aligned}$$

and so

$$\|\mathcal{T}_{u,z_0}(t)\|_{H^{s'}} \le C\|z_0\|_{H^{s'}} + C\varepsilon\delta^3 + C\left(\varepsilon+\delta\right)\delta^2 + C\left(\varepsilon+\delta\right)^2\delta^2.$$

Then, we also have

$$\begin{aligned}
\mathcal{T}_{u,z_0}(t) ={}& e^{-\imath tH + \imath\int_0^t E(u(r))\, dr} z_0\\
&+ \int_0^t e^{-\imath(t-v)H + \imath\int_v^t E(u(r))\, dr}\mathbf{P}_c(u(v))Jd^2F(S(u(v)))z(v)\, dv\\
&+ \int_0^t e^{-\imath(t-v)H + \imath\int_v^t E(u(r))\, dr}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\, dv\\
&+ \int_0^t e^{-\imath(t-v)H + \imath\int_v^t E(u(r))\, dr}\mathbf{P}_c(u(v))\times\\
&\quad \times dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\, dv\\
&- \int_0^t e^{-\imath(t-v)H + \imath\int_v^t E(u(r))\, dr}\times\\
&\quad \times (d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\eta(v)\, dv.
\end{aligned}$$

Hence by Lemma 4.4 and Theorem 1.2, we have

$$
\begin{aligned}
&\|\mathcal{T}_{u,z_0}(t)\|_{B^{\beta}_{\infty,2}}\\
&\le C\langle t\rangle^{-3/2}\|z_0\|_{B^{\beta+3}_{1,2}}+C\int_0^t\langle t-v\rangle^{-3/2}\left\|d^2F(S(u(v)))z(v)\right\|_{B^{\beta+3}_{1,2}}\,dv\\
&\quad+C\int_0^t\langle t-v\rangle^{-3/2}\left\|N(u(v),\eta(v))\right\|_{B^{\beta+3}_{1,2}}\,dv\\
&\quad+C\int_0^t\langle t-v\rangle^{-3/2}\left\|dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\right\|_{B^{\beta+3}_{1,2}}\,dv\\
&\quad+C\int_0^t\langle t-v\rangle^{-3/2}\times\\
&\qquad\times\left\|(d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)),dS(u(v))\rangle\eta(v)\right\|_{B^{\beta+3}_{1,2}}\,dv.
\end{aligned}
$$

With Lemma 5.1, we infer

$$
\begin{aligned}
&\|\mathcal{T}_{u,z_0}(t)\|_{B^{\beta}_{\infty,2}}\le\\
&C\langle t\rangle^{-3/2}\|z_0\|_{B^{\beta+3}_{1,2}}+C\int_0^t\langle t-v\rangle^{-3/2}\left|u(v)\right|^2\left\|z(v)\right\|_{H^{\beta+3}_{-\sigma}}\,dv\\
&+C\int_0^t\langle t-v\rangle^{-3/2}\left(|u(v)|+\|\eta(v)\|_{H^{\beta+3}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|_{H^{\beta+3}}\,dv\\
&+C\int_0^t\langle t-v\rangle^{-3/2}\left(|u(v)|+\|\eta(v)\|_{H^{\beta+3}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|_{H^{\beta+3}}\,dv\\
&+C\int_0^t\langle t-v\rangle^{-3/2}\left(|u(v)|+\|\eta(v)\|_{H^{\beta+3}}\right)\|\eta(v)\|_{L^\infty}\|\eta(v)\|^2_{H^{\beta+3}}\,dv.
\end{aligned}
$$

With the estimate

$$
\int_0^t\langle t-v\rangle^{-3/2}\langle v\rangle^{-3/2}\,dv\le C\langle t\rangle^{-3/2},
$$

we infer

$$
\langle t\rangle^{3/2}\|\mathcal{T}_{u,z_0}(t)\|_{B^{\beta}_{\infty,2}}\le C\|z_0\|_{B^{\beta+3}_{1,2}}+C\varepsilon^2\delta+C\left(\varepsilon+\delta\right)\delta^2+C\left(\varepsilon+\delta\right)^2\delta^2.
$$

Then we also have

$$\begin{aligned}\mathcal{T}_{u,z_0}(t) = e^{-\imath tH+\imath\int_0^t E(u(r))\,dr} z_0 + \int_0^t e^{-\imath(t-v)H+\imath\int_v^t E(u(r))\,dr}\times \\ \times \mathbf{P}_c(u(v))J\nabla F(\eta(v))\,dv \\ + \int_0^t e^{-\imath(t-v)H+\imath\int_v^t E(u(r))\,dr}\times \\ \times \mathbf{P}_c(u(v))J\{\nabla F(S(u(v))+\eta(v)) - \nabla F(S(u(v)) - \nabla F(\eta(v))\}\,dv \\ + \int_0^t e^{-\imath(t-v)H+\imath\int_v^t E(u(r))\,dr}\times \\ \times \mathbf{P}_c(u(v))dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv \\ - \int_0^t e^{-\imath(t-v)H+\imath\int_v^t E(u(r))\,dr}\times \\ \times (d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\eta(v)\,dv.\end{aligned}$$

We now use Lemma 4.4 and Theorem 1.1, except for the second term of the right hand side for which we used Theorem 1.2 since $\sigma > 3/2$. We also use Lemma 5.1, except for the third term of the right hand side for which an obvious adaptation of the proof of Lemma 5.3 gives

$$\begin{aligned}\|\nabla F(S(u(v))+\eta(v)) - \nabla F(S(u(v)) - \nabla F(\eta(v))\|_{H^s_\sigma} \\ \le C\,(|u(v)| + \|\eta(v)\|_{H^s})\,|u(v)|\,\|\eta(v)\|_{H^s_{-\sigma}}\end{aligned}$$

and so we obtain

$$\begin{aligned}\langle t\rangle^{3/2}\|\mathcal{T}_{u,z_0}(t)\|_{H^s_{-\sigma}} \\ \le C\|z_0\|_{H^s_\sigma} + C\delta^3 + C\,(\varepsilon+\delta)\,\delta + C\,(\varepsilon+\delta)\,\delta^2 + C\,(\varepsilon+\delta)^2\,\delta^2.\end{aligned}$$

Therefore we have that $\mathcal{T}_{u,z_0}$ leaves $\mathcal{Z}(u,\delta)$ invariant if $\|z_0\|_{H^{s'}_\sigma}$, δ and ε are small enough.

An important property of $\mathcal{T}$ is given by the

Lemma 5.8. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that there exists $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, for any u, $u' \in \mathcal{U}(\varepsilon,\delta)$, for any $z_0 \in \mathcal{H}_c(u(0))$, for any $z_0' \in \mathcal{H}_c(u'(0))$, for $z \in \mathcal{Z}(u,\delta)$ and for any $z' \in \mathcal{Z}(u',\delta)$, one has*

$$\begin{aligned}\left\|\mathcal{T}_{u',z_0'}(z') - \mathcal{T}_{u,z_0}(z)\right\|_{L^\infty(\mathbb{R}^+,H^{s'})} \\ \le \|z_0 - z_0'\|_{L^\infty(\mathbb{R}^+,H^{s'})} + \kappa\left\{\|u-u'\|_{L^\infty} + \|z-z'\|_{L^\infty(\mathbb{R}^+,H^{s'})}\right\}.\end{aligned}$$

Proof. It is an easy consequences of straightforward estimates on the following identity

$$\begin{aligned}\mathcal{T}_{u,z_0}(t) &= e^{-\imath tH(u_\infty)}z_0 \\ &- \int_0^t e^{-\imath(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left(E(S(u(v))) - E(S(u_\infty))\right)z(v)\,dv \\ &+ \int_0^t e^{-\imath(t-v)H(u_\infty)}\mathbf{P}_c(u(v))J\left(d^2F(S(u(v))) - d^2F(S(u_\infty))\right)z(v)\,dv \\ &+ \int_0^t e^{-\imath(t-v)H(u_\infty)}\mathbf{P}_c(u(v))JN(u(v),\eta(v))\,dv \\ &+ \int_0^t e^{-\imath(t-v)H(u_\infty)}\mathbf{P}_c(u(v))dS(u(v))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv \\ &- \int_0^t e^{-\imath(t-v)H(u_\infty)}(d\mathbf{P}_c(u(v)))A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\eta(v)\,dv,\end{aligned}$$

based only on Lemma 5.3, 5.4 and 5.5 and on the fact that

$$\begin{aligned}&\mathbf{P}_c(u_\infty)\left(e^{-\imath sH(u_\infty)} - e^{-\imath sH(u'_\infty)}\right)\mathbf{P}_c(u(v)) = \\ &-\mathbf{P}_c(u_\infty)\int_0^s\left(e^{-\imath(s-s')H(u_\infty)}(E(S(u'_\infty)) - E(S(u_\infty))\,e^{-\imath s'H(u'_\infty)}\right)ds'\mathbf{P}_c(u(v)) \\ &+\mathbf{P}_c(u_\infty)\int_0^s\left(e^{-\imath(s-s')H(u_\infty)}\Big(d^2F(S(u'_\infty)) - d^2F(S(u_\infty))\Big)e^{-\imath s'H(u'_\infty)}\right)ds'\mathbf{P}_c(u(v))\end{aligned}$$

form a family of operator in $\mathcal{B}(H^{s'}(\mathbb{R}^3,\mathbb{C}^8), H^{s'}(\mathbb{R}^3,\mathbb{C}^8))$ (to this end we use Lemma A.2) such that

$$\left\|\mathbf{P}_c(u_\infty)\left(e^{-\imath sH(u_\infty)} - e^{-\imath sH(u'_\infty)}\right)\mathbf{P}_c(u(v))\right\| \le C\varepsilon\left|u'_\infty - u_\infty\right|,$$

with C independent of u, u'.

Lemma 5.9. *There exists $\delta > 0$ and $\varepsilon > 0$ such that for any $u \in \mathcal{U}(\delta,\varepsilon)$ there is a solution $z \in \mathcal{Z}(u,\delta)$ of the equation*

$$\begin{cases}\partial_t z &= JH(u)z + \mathbf{P}_c(u)JN(u,\eta) \\ &\quad +\mathbf{P}_c(u)dS(u)A(u,\eta)\langle N(u,\eta), dS(u)\rangle \\ &\quad -(D\mathbf{P}_c(u))A(u,\eta)\langle N(u,\eta), dS(u)\rangle\eta, \\ z(0) = z_0,\end{cases} \tag{5.2}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$, whenever $z_0 \in H_\sigma^{s'}$ is small enough.

Proof. It is a consequence of the fix point theorem applied to $\mathcal{T}_{u,z_0}$.

Lemma 5.10. *Under the assumptions of Lemma 5.9, for any $u \in \mathcal{U}(\delta, \varepsilon)$ and solution z of* (5.2)*, with $z_0 \in H^{s'}_\sigma$ small, the following limit*

$$z_\infty = \lim_{t\to\infty} e^{\imath tH - \imath \int_0^t E(u(r))\, dr} z(t)$$

exists in $H^{s'} \cap B^\beta_{\infty,2} \cap H^s_{-\sigma}$. Moreover, we have $z_\infty \in \mathcal{H}_c(0)$ and

$$\|e^{-\imath tH + \imath \int_0^t E(u(r))\, dr} z_\infty - z(t)\|_{H^{s'}} \leq C\delta^2,$$
$$\|e^{-\imath tH + \imath \int_0^t E(u(r))\, dr} z_\infty - z(t)\|_{B^\beta_{\infty,2}} \leq C\frac{\delta^2}{\langle t\rangle^2},$$
$$\|e^{-\imath tH + \imath \int_0^t E(u(r))\, dr} z_\infty - z(t)\|_{H^s_{-\sigma}} \leq C\frac{\delta^2}{\langle t\rangle^2}.$$

Proof. Using exactly the same method as the one of Lemma 5.7, applied to

$$\begin{aligned}
e^{\imath tH - \imath \int_0^t E(u(r))\, dr} z(t) &= z_0 \\
&+ \int_0^t e^{\imath vH - \imath \int_0^v E(u(r))\, dr} \mathbf{P}_c(u(v)) JN(u(v), \eta(v))\, dv \\
&+ \int_0^t e^{\imath vH - \imath \int_0^v E(u(r))\, dr} \times \\
&\times \mathbf{P}_c(u(v)) dS(u(v)) A(u(v), \eta(v)) \langle N(u(v), \eta(v)), dS(u(v))\rangle\, dv \\
&- \int_0^t e^{\imath vH - \imath \int_0^v E(u(r))\, dr} \times \\
&\times (d\mathbf{P}_c(u(v))) A(u(v), \eta(v)) \langle N(u(v), \eta(v)), dS(u(v))\rangle \eta(v)\, dv,
\end{aligned}$$

we prove that the limit exist by the same way we also obtain the convergence rate. Since $e^{-itH} z_\infty$ tends to zero, z_∞ necessarily belongs to $\mathcal{H}_c(0)$.

Remark 5.1. The preceding proof also work with the formula

$$\begin{aligned}
e^{\imath tD_m - \imath \int_0^t E(u(r))\, dr} z(t) &= z_0 + \int_0^t e^{\imath vD_m - \imath \int_0^v E(u(r))\, dr} \mathbf{P}_c(u(v)) V z(v)\, dv \\
&+ \int_0^t e^{\imath vD_m - \imath \int_0^v E(u(r))\, dr} \mathbf{P}_c(u(v)) JN(u(v), \eta(v))\, dv \\
&+ \int_0^t e^{\imath vD_m - \imath \int_0^v E(u(r))\, dr} \times \\
&\times \mathbf{P}_c(u(v)) dS(u(v)) A(u(v), \eta(v)) \langle N(u(v), \eta(v)), dS(u(v))\rangle\, dv \\
&- \int_0^t e^{\imath vD_m - \imath \int_0^v E(u(r))\, dr} \times \\
&\times (d\mathbf{P}_c(u(v))) A(u(v), \eta(v)) \langle N(u(v), \eta(v)), dS(u(v))\rangle \eta(v)\, dv.
\end{aligned}$$

Hence we obtain the same result with

$$e^{-\imath tD_m + \imath \int_0^t E(u(r))\, dr} \tilde{z}_\infty$$

instead of
$$e^{-\imath tH+\imath\int_0^t E(u(r))\,dr}z_\infty.$$
But we obtain the estimates
$$\|e^{-\imath tD_m+\imath\int_0^t E(u(r))\,dr}\widetilde{z}_\infty - z(t)\|_{H^{s'}} \leq C\delta,$$
$$\|e^{-\imath tD_m+\imath\int_0^t E(u(r))\,dr}\widetilde{z}_\infty - z(t)\|_{B^\beta_{\infty,2}} \leq C\frac{\delta}{\langle t\rangle^2},$$
$$\|e^{-\imath tD_m+\imath\int_0^t E(u(r))\,dr}\widetilde{z}_\infty - z(t)\|_{H^s_{-\sigma}} \leq C\frac{\delta}{\langle t\rangle^2}.$$

5.3. Step 3: Construction of u. Here we want to solve the equation for u. We notice that z and α have been built in the previous section and are functions of u and $z_0 \in \mathcal{H}_c(u(0))$. Let us introduce for any $\alpha \in \Omega(\delta)$ and $u_0 \in B_{\mathbb{C}}(0,\varepsilon)$ the function on $\mathcal{U}(\varepsilon,\delta)$:
$$f_{u_0}(u)(t) = u_0 - \int_0^t A(u(v),\eta(v))\langle N(u(v),\eta(v)), dS(u(v))\rangle\,dv,$$
where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$. We have the

Lemma 5.11. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, the function f_{u_0} maps $\mathcal{U}(\varepsilon,\delta)$ into itself if u_0 and $z_0 \in H^{s'}_\sigma \cap \mathcal{H}_c(u_0)$ are small enough.*

Proof. By means of Lemma 5.1, we obtain
$$|f_{u_0}(u)(t)| \leq |u_0| + C\int_0^t \|N(u(v),\eta(v))\|_{H^s_{-\sigma}} \leq |u_0| + C\,(\varepsilon+\delta)\,\delta^2.$$
Hence for u_0 and δ small $f_{u_0}(u)(t) \in B_{\mathbb{C}}(0,\varepsilon)$. Estimate (5.1) also gives the existence of $(f_{u_0}(u))_\infty = \lim_{t\to+\infty} f_{u_0}(u)(t)$ and then
$$|(f_{u_0}(u))_\infty - f_{u_0}(u)(t)| \leq C\int_t^{+\infty} \|N(u(v),\eta(v))\|_{H^s_{-\sigma}} \leq \frac{C}{t^2}\,(\varepsilon+\delta)\,\delta^2.$$
The function f_{u_0} has also a Lipshitz property as stated by the

Lemma 5.12. *There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that there exists $\kappa \in (0,1)$ such that for any $\delta \in (0,\delta_0)$, for any $\varepsilon \in (0,\varepsilon_0)$, for any u, $u' \in \mathcal{U}(\varepsilon,\delta)$, for any $z_0 \in \mathcal{H}_c(u(0)) \cap H^{s'}_\sigma$, for any $z_0' \in \mathcal{H}_c(u'(0)) \cap H^{s'}_\sigma$ small enough, for u_0, u_0' small enough, there exists $\kappa \in (0,1)$ such that*
$$\left|f_{u_0}(u) - f_{u_0'}(u')\right|_{L^\infty(\mathbb{R}^+)} \leq |u_0 - u_0'| + \kappa\left(\|u-u'\|_{L^\infty(\mathbb{R}^+)} + \|z_0 - z_0'\|_{H^{s'}_\sigma}\right).$$

Proof. This a straightforward consequence of Lemma 5.3, 5.4, 5.5 and 5.8.

We are now able to prove the

Lemma 5.13. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ such that for any $u_0 \in \mathbb{C}$ small and $z_0 \in \mathcal{H}_c(u_0) \cap H^{s'}_\sigma$ small, the equation*

$$\begin{cases} \dot{u} &= -A(u,\eta)\langle N(u,\eta), dS(u)\rangle, \\ u(0) &= u_0, \end{cases}$$

where $\eta(t) = \alpha^+(t)S_1^+(u) + \alpha^-(t)S_1^-(u) + z(t)$, has a unique solution in $\mathcal{U}(\delta, \varepsilon)$.

Proof. This is also a straightforward consequence of the fixed point theorem for f_{u_0}.

5.4. Step 4: End of the proof of Theorem 1.3. We now conclude our proof with the

Lemma 5.14 (Decomposition lemma 2). *Let be $s \geq 0$ and $p \geq 1$ there exist $\delta > 0$ and a $\mathcal{C}^\infty$ map $U_0 : B_{W^{s,p}}(0,\delta) \mapsto B_{\mathbb{C}}(0,\varepsilon)$ which satisfies for $\psi \in B_{W^{s,p}}(0,\delta)$*

$$\psi = S(u) + \eta, \text{ with } \eta \in \{\phi_0\}^\perp \Longleftrightarrow u = U_0(\psi)$$

Proof. In fact, we just write ψ with respect to the spectral decomposition of H:

$$\psi = u\phi_0 + r = S(u) + \eta$$

where $r \in \{\phi_0\}^\perp$ and $\eta = r - h(u)$, with h defined in Proposition 1.1.

With respect to the notation of Theorem 1.3 for $\psi_0 = S(u_0) + \alpha^+(0)S_1^+(u_0) + \alpha^-(0)S_1^-(u_0) + \mathbf{P}_c(u_0)\widetilde{z_0}$ where $\widetilde{z_0} \in \mathrm{Ran}(\mathbf{P}_c)$, we introduce

$$\begin{cases} v_0 = U_0(\psi_0) \\ \xi_0 = \mathbf{P}_c\left(\psi_0 - S(v_0)\right) \end{cases}$$

and

$$\begin{cases} G(u_0,\widetilde{z_0})_1 = U_0(S(u_0) + \alpha^+(0)S_1^+(u_0) + \alpha^-(0)S_1^-(u_0) + \mathbf{P}_c(u_0)\widetilde{z_0}) \\ G(u_0,\widetilde{z_0})_2 = \mathbf{P}_c\Big(S(u_0) - S(G(u_0,\widetilde{z_0})_1) \\ \qquad\qquad\qquad +\alpha^+(0)S_1^+(u_0) + \alpha^-(0)S_1^-(u_0) + \mathbf{P}_c(u_0)\widetilde{z_0}\Big) \end{cases}$$

Then using $U_0(S(u_0)) = u_0$, we write $G(u_0,\widetilde{z_0}) = (u_0,\widetilde{z_0}) + \widetilde{G}(u_0,\widetilde{z_0})$, with

$$\left\|\widetilde{G}(u_0,\widetilde{z_0}) - \widetilde{G}(u_0',\widetilde{z_0}')\right\|_{H^{s'}_\sigma} \leq \kappa\left(|u_0 - u_0'| + \left\|\widetilde{z_0} - \widetilde{z_0}'\right\|_{H^{s'}_\sigma}\right)$$

with $\kappa \leq 1/2$ if u_0, u_0' and $\widetilde{z_0}, \widetilde{z_0}'$ small enough. Hence in this case G is invertible with a Lipshitz inverse F. Then we choose

$$\begin{aligned} &\Psi(v_0,\xi_0) \\ &\quad = \big\langle S(F(v_0,\xi_0)_1) \\ &\quad +\alpha(F(v_0,\xi_0)_1)^+(0)S_1^+(F(v_0,\xi_0)_1) + \alpha(F(v_0,\xi_0)_1)^-(0)S_1^-(F(v_0,\xi_0)_1) \\ &\quad +\mathbf{P}_c(F(v_0,\xi_0)_1)F(v_0,\xi_0)_2 - S(v_0), \phi_1\big\rangle \end{aligned}$$

and

$$\xi_\infty = (\mathbf{P}_c(F(v_0,\xi_0)_1)F(v_0,\xi_0)_2)_\infty$$

and

$$E_\infty = \int_0^\infty \{E(F(v_0,\xi_0)_1(v)) - E\left((F(v_0,\xi_0)_1)_\infty\right)\}\ dv.$$

In the proof of Lemma 5.7, we see that δ is of the same order as $\|\xi_0\|_{H^{s'}_\sigma}$. The rest of the Theorem easily follows.

$\square$

APPENDICES

A. The wave operator and similarity for the linearized operator

Inspired by [Kat66], we use an argument of similarity to prove the

Lemma A.1. *For all $s \in \mathbb{R}^+$, there exists $C_s > 0$, such that*

$$\forall t \in \mathbb{R},\ \|e^{tJH(z)}\|_{\mathcal{L}(H^s)} \le C_s.$$

We prove this lemma by using the boundedness in H^s of the wave operator:

$$W_\pm = s - \lim_{t\to\pm\infty} e^{-tH(z)^*J}e^{-\imath t(H-E(z))}\mathbf{P}_c(H)$$

and the intertwining property:

$$e^{-tH(z)^*J}\mathbf{P}_c(z)^* = W^\pm e^{-\imath t(H-E(z))}\mathbf{P}_c(H)(W^\pm)^{-1}.$$

This boundedness follows from the

Lemma A.2 (Smooth and small non-selfadjoint perturbations). *Let be $\psi \in L^2$ and $\sigma \ge 1$. Then there exists $\varepsilon > 0$ and $C > 0$ such that*

$$\forall z \in B_{\mathbb{C}}(0,\varepsilon),\ \int_0^\infty \| < Q >^{-\sigma}\ e^{sJH(z)}\mathbf{P}_c(z)\psi\|_2^2\ ds \le C\|\psi\|_2^2. \qquad \text{(A.1)}$$

Proof. By Lemma (4.4), we prove $\mathbf{P}_c(z)R(z,0)\mathbf{P}_c(0)\mathbf{P}_c(z) = \mathbf{P}_c(z)$. Taking the adjoint with respect to the real structure, we infer $\mathbf{P}_c(z)\mathbf{P}_c(0)R(z)^*\mathbf{P}_c(z) = \mathbf{P}_c(z)$. Then, we write

$$\begin{aligned}
&\|\langle Q\rangle^{-\sigma}e^{tJH(z)}\mathbf{P}_c(z)\|\\
&= C\|\langle Q\rangle^{-\sigma}\mathbf{P}_c(z)e^{tJH(z)}\mathbf{P}_c(0)R(z)^*\mathbf{P}_c(z)\|\\
&\le \|\langle Q\rangle^{-\sigma}\mathbf{P}_c(z)e^{-\imath t(H-E(z))}\mathbf{P}_c(0)R(z)^*\mathbf{P}_c(z)\|\\
&\quad+\int_0^t\|\langle Q\rangle^{-\sigma}\mathbf{P}_c(z)e^{(t-s)JH(z)}D\nabla F(S(z))e^{-\imath s(H-E(z))}\mathbf{P}_c(0)R(z)\mathbf{P}_c(z)\|\,ds\\
&\le \|\langle Q\rangle^{-\sigma}\mathbf{P}_c(z)e^{-\imath t(H-E(z))}\mathbf{P}_c(0)R(z)\mathbf{P}_c(z)\|\\
&\quad\quad+C|z|^2\int_0^t \|\langle Q\rangle^{-\sigma}\mathbf{P}_c(z)e^{(t-s)JH(z)}\langle Q\rangle^{-\sigma}\|\|\langle Q\rangle^{-\sigma}e^{-\imath s(H-E(z))}\mathbf{P}_c(0)\|\,ds.
\end{aligned}$$

Using Proposition 3.10, we obtain the claim (A.1) for z sufficiently small.

This give us the existence and the boundedness of the wave operator, as stated by the following

Lemma A.3. *Let be* $W_t = e^{-tH(z)^*J}e^{-it(H-E(z))}\mathbf{P}_c(H)$*. Then the limits*

$$W^{\pm} = \lim_{t\to\pm\infty} W_t$$

exist in $B(H^s)$ *and their range is* $\mathrm{Ran}\,(\mathbf{P}_c(z))$*. The same is true for* W_t^* *and*

$$\left(W^{\pm}\right)^{-1} = \lim_{t\to\pm\infty} (W_t)^{-1}.$$

Proof. Let us define $W_t = e^{-tH(z)^*J}e^{-it(H-E(z))}$. we have for $\phi \in H_c(z)$ and $\psi \in H_c(0)$

$$\langle \phi, W_t\psi\rangle = \langle \phi, \psi\rangle + \int_0^t \left\langle \phi, \frac{d}{ds}W_s\psi \right\rangle ds,$$

Since we have

$$\begin{aligned}\left\langle \phi, \frac{d}{ds}W_s\psi \right\rangle &= \left\langle e^{-tJH(z)}\phi, D\nabla F(S(z))e^{-\imath t(H-E(z))}\psi \right\rangle \\ &\leq C|z|^2 \|\langle Q\rangle^{-\sigma}e^{tJH(z)}\phi\|\|\langle Q\rangle^{-\sigma}e^{-\imath t(H-E(z))}\psi\|.\end{aligned}$$

which gives $\langle \phi, \frac{d}{ds}W_s\psi\rangle \in L^1(\mathbb{R})$, so $W_{\pm}$ exists and is bounded in $\mathcal{L}(H_c(0), H_c(z))$ by the previous lemma. Since for any vector ϕ in an eigenspace of $JH(z)$, $W_t^*\phi$ tends weakly to zero, we obtain that the range of $W^{\pm}$ is a subspace of the range of $\mathbf{P}_c(z)$. Then the same statements about $(W_t)^{-1}$ follows by the same way. The invertibility is then immediate.

Proof (of Lemma A.1). The L^2 bound follows from the intertwining property as explained before Lemma A.2.
The proof of the H^k bounds follows from commutation argument, we apply the same scheme to

$$\begin{aligned}\partial_i e^{-tJH(z)}\mathbf{P}_c(z) &= [\partial_i, \mathbf{P}_c(z)]e^{-tJH(z)}\mathbf{P}_c(z) + \mathbf{P}_c(z)e^{-tJH(z)}[\partial_i, \mathbf{P}_c(z)] \\ &\quad + \mathbf{P}_c(z)[\partial_i, e^{-tJH(z)}]\mathbf{P}_c(z) \\ &= [\partial_i, \mathbf{P}_c(z)]e^{-tJH(z)}\partial_i + \mathbf{P}_c(z)e^{-tJH(z)}[\partial_i, \mathbf{P}_c(z)] \\ &\quad + \int_0^t e^{-(t-s)JH(z)}\mathbf{P}_c(z)(\partial_i D\nabla F(S))e^{-sJH(z)}\mathbf{P}_c(z)dz.\end{aligned}$$

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